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Minimum cycle bases of products of Fuzzy graphs

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Abstract: In this paper we extend the concept of a minimum cycle basis of a graph for the fuzzy graphs from the minimum length cycle bases of the factors. We also apply the concept for the Cartesian product of fuzzy graph. This paper will basically helpful for the researchers who are working on genetics, i.e. restructuring of DNA cycle, which we assume is a product of Protein chain.

Keywords: Product of fuzzy graph; fuzzy vector space; minimum cycle bases.

I. INTRODUCTION

Minimum length bases of the cycle space of a graph (MCBs)which attracts the researchers due to its applications in the various areas of Science and technology. In 1847 G. Kirchhoff first presents a treatise on electrical networks [4]. Later Berger, Flamm,Gleiss, Leydold, and Stadler (2004) describe an application of minimum cycle bases to the problem. Those researchers were applied this concept in chemical information systems, structural flexibility analysis, the force method of frame analysis, etc but they applied the concept for crisp graph. In this paper we modify the concept for fuzzy graph which will applicable for all the areas of previous work and more in the field of genetics, Optimization techniques and Network analysis etc. In the article *Minimum Cycle Bases of Product Graphs* [2],W. Imrich and P. Stadler construct minimum cycle bases for Cartesian and strong products for crisp graphs. F. Berger solves the same problem for the lexicographical product for crisp graph [3].Keeping aforesaid application in our mind, in this paper we discuss about the minimum cycle bases in Cartesian product of fuzzy graph. Which may have an application in the field of genetics for example, suppose a dark complexioned couple wishes to have a fair complexioned baby? It depends upon the DNA structure of the baby which is the product of the DNA helices of the couple. Suppose in one of the couple's family there was a white skinned person, then the membership of the component of DNA which effects the color of skin becomes high, in that particular couple. Then our task is to choose a minimum cycle basis of the components of DNA so that product will generate a cycle which produces the desired result which is nothing but only to find a Minimum length bases of the cycle space of the product of a fuzzy graph i.e., MCBs of a fuzzy graph. The motivation towards this work is obvious because of its huge applications in the field of network optimization and analysis as for example for multi connected network it is very difficult to analyse the network at any stage, so if we observe the normal graph by vector space of the graph then it becomes easy to analyze it.

II. PRELIMINARIES

*Definition 2.1.*A fuzzy set V is a mapping σ from V to [0, 1]. A fuzzy graph G is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ, i.e. $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$. The underlying crisp graph of G = (σ, μ) is denoted by G^*= (V, E) where $E \subseteq V \times V$.

Definition 2.2. A pair $\tilde{V}$= (V; μ) is said to be a fuzzy vector space, when V is a vector space over a field F, and μ: V →[0; 1] is a mapping which satisfy the condition μ (kx + ly) $>$ μ (x) $\wedge\mu$ (y) for any x, y ∈V and k, l ∈F.

*Definition 2.3.*Let $\tilde{V}$= (V, μ) be a fuzzy vector space. A set B = $\{s_1, s_2, s_3, \ldots\ldots\ldots s_n\}$ of vectors is said to be a basis of $\tilde{V}$, if the following statements are satisfied:

(i) B is a basis of V;
(ii) For any $\{a_i\}_{i=1}^{n} \subseteq$ F we have
$\mu(\sum_{i=1}^{n} a_i s_i) = \bigwedge_{i=1}^{n} \mu(a_i s_i)$we denote all bases of $\tilde{V}$ by $\mathfrak{B}$(V).

*Definition 2.4.*The Cartesian product G = $G_1 \times G_2$ of two fuzzy graphs G_1 = (A_1, B_1) and G_2 = (A_2, B_2) of the underlying crisp graphs G^*_1 = (V_1, E_1) and G^*_2 =(V_2, E_2) is defined as the pair ($A_1 \times A_2$, $B_1 \times B_2$) where A_1 and A_2 are fuzzy set of vertices and B_1 and B_2 are fuzzy sets for edges such that:

(i) $(M_{A1}\times M_{A2})(u_1, u_2) = \min(M_{A1}(u_1), M_{A2}(u_2))$ *for all* $(u_1, u_2) \in V$.

(ii) $(M_{B1}\times M_{B2})((u, u_2)(u, v_2)) = \min(M_{A1}(u), M_{B2}(u_2v_2))$ *for all* $u\in V_1$ *and* $u_2v_2\in E_2$.

(iii) $(M_{B1}\times M_{B2})((u_1, z)(v_1, z)) = \min(M_{B1}(u_1v_1), M_{A2}(z))$ *for all* $z\in V_2$ *and* $u_1v_1\in E_1$.

III. THE CYCLE SPACE OF A FUZZY GRAPH

For a given Fuzzy graph G(σ, μ), let $\mathcal{E}$(G) be the power set of a fuzzy set E(G) including the empty set with membership value μ. This makes a fuzzy vector space over the two-element field GL(2)= {0,1}. Where we denote zero vector by θ=0 and sum of two vectors X, Y $\in \mathcal{E}(G)$ is fuzzy symmetric difference between X and Y. For an edge space we define a map $f: \mathcal{E}(G) \to \mathfrak{V}(G)$ s.t. $f\{(x, \sigma_1), (y, \sigma_2)\} = (x + y, \min(\sigma_1, \sigma_2))$. We call $\mathcal{E}$(G) the edge space of a fuzzy graph. Therefore E(G) is a basis for the edge space $\mathcal{E}$(G) and its dimension is dim($\mathcal{E}$(G))=|E(G)|.

Similarly, vertex space $\mathfrak{V}$(G) of a fuzzy graph G is obtained by taking the power set of V(G) and viewed as fuzzy vector space over the field GL(2) when vector sum is taken as fuzzy symmetric difference between any two sets of the power set of V(G). The sub-space $\mathbb{C}(G)$= Ker(f) is called the cycle space of the fuzzy graph G and ker(f) contains the cycles which are obtained by adding any independent edges to the spanning forest of G and it also contains the linear combinations of all cycles.

*Definition3.1.*The independent cycles of the cycle space is called cycle basis for a fuzzy graph G if it generates all other possible cycles of the space. The minimum length cycle basis which span all other cycles is called Minimum Cycle Basis or MCB.

The *cycles C's* are subgraph of *G* whose all vertices have even degree. They are *simple* if connected and if each vertex in *C* have degree two. A cycle *C* can be represented by incidence vector $(\lambda_i(C))_{i=1,\dots,m}$, with

$$\lambda_i(C) = \begin{cases} 0, & \text{if } e_i \notin E(C), \\ 1, & \text{if } e_i \in E(C), \end{cases}$$

where *E(C)* represents the set of edges in *C*. Generally, we identify a cycle *C* by its incidence vector. The membership of a cycle *C* is defined as $\omega(C) = \sum_{e\in C} \omega(e)$.

PROPOSITION I. *If a fuzzy graph G has m number of edges, n number of vertices and* c(G) *number of connected components than the dimension d of* $\mathbb{C}_{GF(2)}(G)$ *is equal to* $m - n + c(G)$.

Since, we assume that G is a simple graph so for cycle basis of G let F be a maximal spanning forest of G, So the set S=E(G) – E(F) contains the edges such that on F+e it gives the independent cycle. Hence the number of independent cycle i.e., dimension of $\mathbb{C}_{GF(2)}(G)$= *E(G)-E(F)*=E(G) – (V(G) – c(G)). So, $d = m - n + c(G)$.

We called d, cyclomatic number of *G*. A *cycle basis B* is a basis of *GF(2)(G)*. It is called a *minimum cycle basis* if its membership $\omega(B) = \bigwedge_{c\in B}\omega(C)$ is minimum and intersection of the membership of all edges of a cycle is called cyclomatic membership.

*Exampal***I.** Let G be a fuzzy graph with five vertices and six edges shown in the figure1. Each edges and vertices assigns some membership value, now we have to find MCB of G.

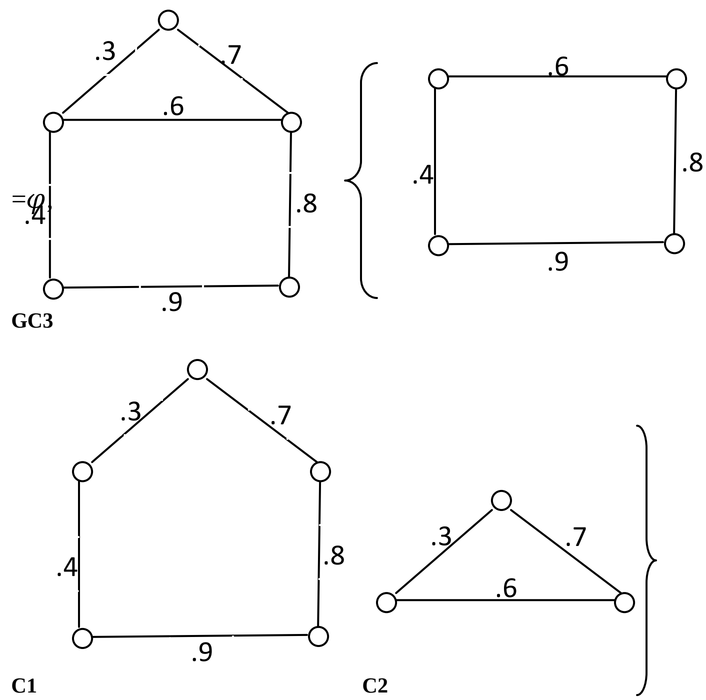


Fig I**:** Fuzzy graph G and possible cycle basis

In Fig I, for graph G first we obtained the cycles, which we obtained by joining the independent edges to the minimum strength spanning tree. Here the minimum strength spanning tree contains the edges having membership value .3, .6, .4 and .9. Thus we get three fundamental cycles and among these three the cycle, C1 and C2 have minimum strength. Hence it remains in the basis. Also according to proposition 1 the dimension of G is 6-5+1=2. Since the cyclomatic membership of C1 and C2 is .3 but for C3 it is .4 and min{ $\omega(C1), \omega(C2), \omega(C3)$}= .3. Hence C1 and C2 are MCB of G.

PROPOSITION 2.Let e∈E be an edge of a fuzzy graph G through which there exists a shortest cycle C(e) with minimum membership. Then a minimum cycle basis B of G must contain C(e).

Moreover, through 'e' it must contain some shortest cycle whose strength is minimum. In general, the set {C(e) : C(e) is a shortest cycle through e∈E} but it does not span $\mathfrak{C}_{GF(2)}(G)$. For an example, in Figure 1 every edge e belongs to one of the four triangles, but according to Proposition 1, dim $\mathfrak{C}_{GF(2)}(G) = 12 - 8 + 1 = 5$.

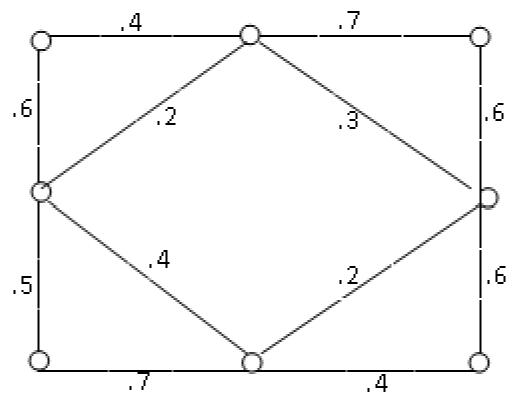


Fig II: Example showing that Proposition 2 generally does not provide a cycle basis.

It is possible to remove the loops and multiple edges form a graph, after removing it, we can choose the minimum cycle basis of the remaining graph, and add to it the cycles corresponding to loops. The basis obtained is a minimum cycle basis of *G*.

Let *u*, *v* be two vertices joined by *k* parallel edges with $k \geq 2$. In such case select an edge e' with minimum membership value among them. For each edge *e* of the remaining $k - 1$ parallel edges compute successively a shortest cycle through *e* and then remove *e* from *G*. The remaining graph after applying this process for all pairs of adjacent vertices is simple. We show that the set of cycles ***B***obtained by computing a minimum cycle basis $\boldsymbol{B}_0$ after reduction of G to simple graph together with the set of cycles $\boldsymbol{B}_1$ obtained from the parallel edges and loops mentioned above form a minimum cycle basis of *G*.

IV. In this section we described an algorithm to find a minimum cycle basis of an undirected fuzzy graph *G*. First, we can assume here that *G* is simple and connected. It depends on the observation that a cycle basis is minimal when no cycle in it may replaced by some another smaller cycle. Thus, the algorithm starts with a fundamental tree basis having minimum membership value and after that if possible it successively exchanges cycles for smaller ones. First we construct specific shortest paths in an auxiliary graph selecting the minimum membership value and then proceed for next. The basic idea is given in Algorithm.

Algorithm for Minimum Cycle Basis

Input: Undirected connected simple fuzzy graph *G*

Output: Minimum cycle basis *B* of *G*

1: Construct a fundamental tree basis $\boldsymbol{B}_0 = \{T_1, \ldots, T_\gamma\}$.

2: **for** $i = 1$ to γ **do**

3: Find the membership $\omega(T_1), \ldots, \omega(T_\gamma)$*and calculate* the minimum.

4: Find a shortest cycle $C_i$which is linearly independent of $\boldsymbol{B}_{i-1} \setminus \{T_i\}$

with Subroutine 2.

5: **if** $\omega(C_i) < \omega(T_i)$ **then**

6: $B_i := (B_{i-1} \setminus \{T_i\}) \in \{C_i\}$

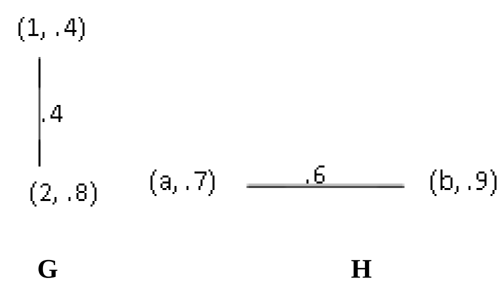
7: **end if**

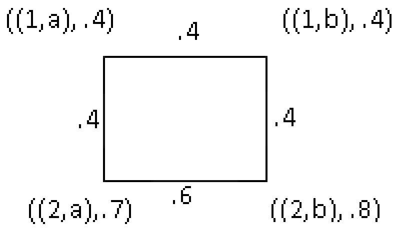
8: **end for**

9: Output $B := B_\gamma$.

V. MCB IN CARTESIAN PRODUCT OF A FUZZY GRAPH

The edge space in the Cartesian product of two fuzzy graph G and H is $\mathcal{E}(G \times H)$ ={(g, h)(g′, h') | if g=g' and e(hh')∈ H or h=h' and e(gg')∈G} and the membership value of each edges are either *min* {σ(g), μ(hh')} or *min* {σ(h), μ(gg')}. Now for the cycle basis of these product graphs we take the all possible sets of forests of G×H. Then the minimum length independent cycles whose cyclomatic membership is minimum remains in the basis.





G×H

Fig: III. Cartesian product of G and H

So, the cycle basis consists those cycles whose membership sum in the forest is minimum.

Induced fuzzy sub graphs in G×H: In G×H the induced sub graph is denoted by ^xG and H^y and is defined as ^xG= G×H[(x, y)| y ϵV_H and μ_y=min{ μ_i| iϵ V_H}] and H^y= G×H [(x, y)| x ϵV_G and μ_x=min{ μ_j| jϵ V_G}] which are fibers of G×H. The edge of G×H can labeled as f^x when μ_{fx}= min{μ_f and σ_x | f ϵ E_H and xϵV_G} and as e^y when μ_{ey} = min{μ_e and σ_y | e ϵ E_G and y ϵV_H}. So, we have e×f={e^x, f^v, e^y, f^u}where e=uvϵ E(G) and f=xyϵ E(H). Two edges are parallel if and only if they are in the form of e^x and e^y or f^v and f^u, respectively.

In a minimal basis of $\mathcal{C}(G \times H)$ corresponding cycles in different fibers can be transformed into each other by a series of ⊕-additions of squares taken from the set *C*.

Lemma:Let C $\epsilon\mathcal{C}(G)$ such that *ω(C)= min{ω(Ci)}and y, z* $\in V_H$. *Then there is a collection of squares* $\mathcal{S} \subseteq \mathcal{C}$(G×H) *such that* $C^z= C^y \oplus \mathcal{S}$.

Proof: For $y, z \in V_H$ let P be a path from y to z consists the edges $e_1, \ldots, e_l$ and the vertices $v_0 = y, v_1, \ldots, v_l= z$. Then $(P^x, \omega(P^x))$ is the corresponding path in the fiber H^x with vertices (x, v_j) and edges(^xe_j, μ). For each edge $g = x_1x_2 \in E_G$ *such that* μ (x_1x_2) is minimum and each path *P* in *H* we write

$C(g; P) = \{^{x1}e_1, ^{x1}e_2, \ldots, ^{x1}e_l, g^z, ^{x2}e_l, ^{x2}e_{l-1}, \ldots, ^{x2}e_1, g^y\}$ *and*

$\omega(C(g; P))=min\{ ^{x1}e_1, ^{x1}e_2, \ldots, ^{x1}e_l, gz, or\ ^{x2}e_l, ^{x2}e_{l-1}, \ldots, ^{x2}e_1, g^y \}$

for the cycle composed of the paths ($^{x1}P, \omega(^{x1}P)$) from (x_1, y) to (x_1, z) and $(^{x2}P, \omega(^{x2}P))$ from (x_2, z) to (x_2, y) together with the edges $(x_1, z)(x_2, z)$ and $(x_2, y)(x_1, y)$.If *P* consists of a single edge *h* we have $C(g;P) = g\times h$. Let P_k and P^k for the subpaths from y to v_k and from v_k to z, respectively. Then

$C(g; P) = C(g, P_k) \oplus C(g, P^k)$

since $C(g,P_k)$ and $C(g,P^k)$ have exactly the edge $(^{vk}g, \mu_{vkg})$ in common. Thus we can decompose any cycle of the form $C(g; P)$ into a ⊕-sum of 4-cycles:

$C(g; P) = \oplus_{j=1}^{l} C(g; e_j) = \oplus_{j=1}^{l} (g\times e_j)$

Now consider a path $\mathcal{S}$ in *G* from *u* to *v* with edges (g_i, μ_{gi}) . Set $C(\mathcal{S}; P) = {^uP} \cup \mathcal{S}^z \cup {^vP} \cup \mathcal{S}^y$ *and* $\omega(C(\mathcal{S}; P))=min\{max\ \mu_{uP}, max\ \mu\, \mathcal{S}_z, max\ \mu_{vP}, max \mu \mathcal{S}_y\}$. Then $C(\mathcal{S}; P) = \oplus i\ C(g_i; P)$ because $C(g_i; P)$ and $C(g_{i-1}; P)$ have the edges of the path ^{xi}P in common, where $x_i = g_{i-1} \cap g_i \in V_G$.

Finally, let $y, z \in H$, $C \in \mathcal{C}(G)$ and *P* a path in *H* connecting *y* and *z*. Let *u* and *v* be adjacent vertices in *C* and write *g* = *uv*. Then $C = \mathcal{S} \cup \{g\}$ where $\mathcal{S}$ is apath in *G* connecting *u* and *v*. Now $C^z \oplus C(\mathcal{S}; P) \oplus C(g, P) = C^y$. We know that both $C(\mathcal{S}; P)$ and $C(g,P)$ can be written as ⊕-sums of squares from $\mathcal{C}_\times$. Similarly, using same technique we can show that for $D \in C(H)$ and $u, v \in V_G$ thereis a collection of squares $\mathcal{S}'' \subseteq \mathcal{C}_\times$ such that $^vD= {^uD} \oplus \mathcal{S}''$.

Theorem I:Let G and H are two fuzzy graphs having maximal spanning forests U⊆G and W⊆H. Then the set $\mathcal{B} = \{g \times h |\ g\epsilon E(G)\ and\ \mu_g = \min\{\mu_i \forall\ i\epsilon\ E(G)\}, h\epsilon E(W)\} \cup \{g \times h |\ g\epsilon E(U), h\epsilon E(H) \setminus E(W)\ and\ \mu_h = \min\{\mu_j\ \forall\ j\epsilon E(H) \setminus E(W)\}\}$ is linearly independent in $\mathcal{S}(G \times H) \subseteq \mathcal{C}(G \times H)$.

Proof: For our convenience let B_L and B_R be the set, left and right side of the union of $\mathcal{B}$. We have to show that B_L and B_R are linearly independent and span (B_L)∩ span (B_R)= {0}.

Let B_L={g_k×h_k | kϵ K} so for the independence of B_L we have to show that $\sum_{k\epsilon K} g_k \times h_k \neq 0$ and $\omega(g_k \times h_k) \neq 0$.

Now take a pendant edge xy of the sub forest such that μ_{xy} is minimum in $\bigcup_{k\epsilon K} h_k \subseteq$ W, such that y is an end vertex.

Therefore, g_k× xy have an edge in the G-layer $p_H^{-1}(b)$. So each g_k× xy contributes a unique such edge. Hence $\sum_{k\epsilon K} g_k \times h_k \neq 0$ because $\sum_{k\epsilon K} g_k \times h_k$ contains an edges in G-layer.

Similarly, we can prove that B_R is linearly independent and any nontrivial linear combinations of its edges have an edge in H-layer.

Let $\sum_{k\epsilon K} g_k \times h_k$= M ϵ Span (B_L) ∩ Span (B_R). If M is non zero then from above we can guarantee that M has an edge {x}×h in H-layer. Then from the definition of B_L and B_R it imply that h ϵ E(W) and $h\epsilon E(H)\setminus E(W)$. So, M =0

Theorem II**:** MCB in the fuzzy leader graph $P_m \times P_n$is the ⊕-sum of independent squares such that σ_x is minimum in P_n and $\sigma_y$is minimum in P_m.

*Proof:*We know that $P_m \times P_n$contains the fiber ${}^xP_m$and ${}^yP_n$ so the membership of ${}^xP_m$= min{σ_x, μ_{gk}}where x$\epsilon V_{Pn}$and $g_k \epsilon E_{Pm}$ similarly, the membership of ${}^yP_n$= min{σ_y, μ_{hk}}where y$\epsilon V_{Pm}$and $h_k \epsilon$ E_{Pn}.. Hence the membership $\omega({}^xP_m \oplus {}^yP_n)$ *is minimum* among all the squares of $P_m \times P_n$. we know from above that squares obtained from product graph is linearly independent.

Let B_L={$g_k \times h_k$ | kϵ K and $g_k \epsilon {}^xP_m$ and $h_k \epsilon$ ${}^yP_n$} span $P_m \times P_n$,andg_k and h_k are independent edges belongs to P_m and P_n respectively such that μ_{g_k} and μ_{hk} are minimum.

Algorithm for MCB in Cartesian product of a fuzzy graph:

In this section we try to find an algorithms for the minimum cycle basis in Cartesian product of a fuzzy graph. For this we first find the minimal spanning forest of a fuzzy graph in such a way that the maximum number of edges in that spanning forest have minimum membership value, we omitted those edges which form a cycle in the spanning forest. Atleast one edge must contained in the spanning forest which has least membership value in the fuzzy graph.

Input: Undirected connected simple fuzzy graphs *G and H*

Output: Minimum cycle basis *B* of *G*×*H*

1: Construct a fundamental tree basis $\boldsymbol{T}_G$ = {T_1, . . . ,T_γ}of G and $\mathbf{T}_H$={L_1,….., L_β}of H.

2: **for** *i*= 1 to *γ and j=1 to β* **do**

3: Find the membership of $\omega(T_1)$,….., $\omega(T_\gamma)$ *and* $\omega(L_1)$,….., $\omega(L_\beta)$ *then* findthe minimum $\omega(T_i)$ and $\omega(L_j)$

4: Find $H_1 = \{e \times f \mid e \in T_G, f \in T_H\}$

$$H_2 = \{C^y \mid C \in B_G, y \in V_H\}$$

$$H_3 = \{{}^xC \mid x \in V_G, C \in B_H\}$$

5: Find $H_1 \cup H_2 \cup H_3$ for subroutine 2 and name it C_i.

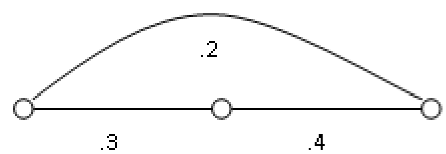
6: Find $\omega(C_i)$, *if* $\omega(C_i) \leq \omega(T_i \cup L_j)$ *then* $C_i \epsilon B$.

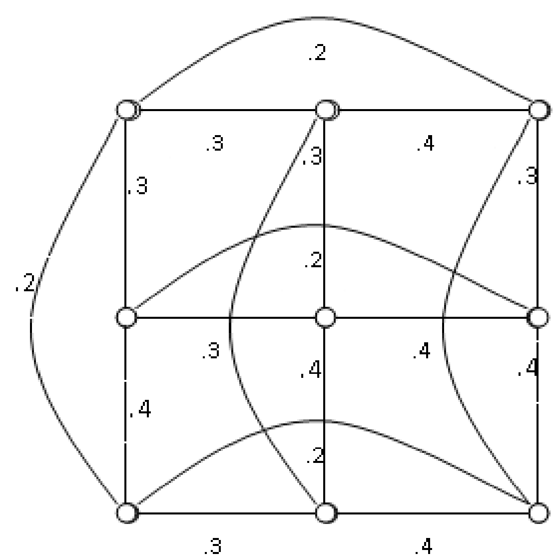
7: **end if** i= *γ or j= β respectively*.

*8:***end for**

9: Output $B := B_{\gamma\beta}$.

*Example:*Let G1 be the K3 fuzzy graph, and let product G1×G1 be its Cartesian product then the MCB of G1×G1 is obtained as follow:





G_1

$G_1 \times G_1$

Possible spanning tree for G1×G1 with minimum weight is :

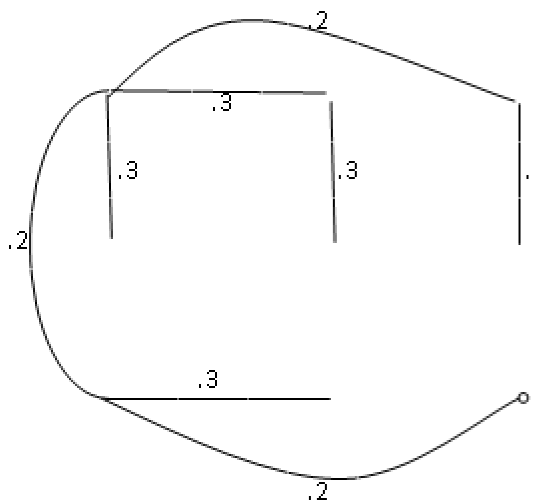


Fig.4. Minimal weighted spanning tree of $G_1 \times G_1$

Hence in above tree when we join independent edges then we get cycles of different length. So on choosing the minimal length minimum weighted cycles we get the MCB which contains three cycles of length three, seven cycles of length four.

VI. CONCLUSION

In this paper we discuss about the minimum cycle bases of the Cartesian product of fuzzy graphs these concepts will be helpful to extend it for the more other important products. These concepts will may extended upto a level which will moderate the sectors like Computer Science, Operations Research, Electrical and Communication, and more other sectors of Engineering in new direction.

VII. REFERENCES

[1] Imrich, W. and Klavzar, S. Product Graphs; Structure and Recognition, Wiley Interscience Series in Discrete Mathematics and Optimization, New York, 2000.

[2] Imrich, W. and Stadler, P. Minimum cycle bases of product graphs, Australasian J. Comb. 26 (2002), 233–244.

[3] Berger, F. Minimum cycle bases of graphs, Dissertation, TechnischeUniversitätMünchen, 2004.

[4] Kirchhoff, G. Uber die Auflösung der Gleichungen, auf welchemanbei der Untersuchung der linearenVertheilunggalvanischerStrömegefürtwird, Annalen der Physik und Chemie 72 (12)(1847) 497–508.

[5] Hammack, R. Minimum cycle bases of direct products of bipartite graphs, Australasian J. Comb. 36 (2006), 213–221.Forman, G. 2003. An extensive empirical study of feature selection metrics for text classification. J. Mach. Learn. Res. 3 (Mar. 2003), 1289-1305.

[6] Hammack, R. Minimum cycle bases of direct products of complete graphs, Information Processing Letters 102 (4) (2007), 214–218.

[7] Bradshaw, Z. and Jaradat, M. M. M. Minimum cycle bases for direct products of K2 with complete graphs, Australasian J. Comb., 43 (2009), 127–131.

[8] Mishra, S. N. and Pal, A. Product of Interval Valued Intuitionistic fuzzy graph, Annals of Pure and Applied Mathematics, 4(2), 2013, 138-144.

[9] Sunitha, M. S. and Vijayakumar, A. A characterization of fuzzy trees, Information Sciences 113 (1999) 293–300.

[10] Sunitha, M. S. and Vijayakumar, A. Some metric aspects of fuzzy graphs, in: R. Balakrishna, H.M. Mulder, A. Vijayakumar (Eds.), Proceedings of the Conference on Graph Connections CUSAT, Allied Publishers, Cochin, 1999, pp. 111–114.

[11] Sunitha, M. S. and Vijayakumar, A. Complement of a fuzzy graph, Indian Journal of Pure and Applied Mathematics 33 (2002) 1451–1464.

[12] Sunitha, M. S. and Vijayakumar, A. Blocks in fuzzy graphs, The Journal of Fuzzy Mathematics 13 (2005) 13–23.