ISSN:2320-0790

An International Journal of Advanced Computer Technology

Some New Classes of Statistical Convergent Fuzzy Real-valued Triple Sequences

Sangita Saha[1], Bijan Nath[2] and Santanu Roy [3*]

[1,2,3] Department of Mathematics, National Institute of Technology, Silchar; Assam, India

Abstract: In this article, some new classes of statistical convergence of sequences of fuzzy real numbers have multiplicity greater than two is introduced. Certain Theorems regarding uniqueness of limit, algebraic characterization of statistical limit for triple sequences of fuzzy numbers are obtained. The decomposition theorem is proved. The inclusion relations are derived. The fuzzy real-valued Cesáro summable triple sequence space is also introduced. A relation between strongly p-Cesáro summability and bounded statistically convergent triple sequences has been established.

Key words and phrases: Density, triple sequence, statistical convergence, statistically Cauchy, fuzzy real-valued triple sequences, Cesáro summable, strong Cesáro summability.

1. Introduction

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [35] in 1965 and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets. In fact the fuzzy set theory has become an area of active area of research in science and engineering for the last 46 years. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in the field of science and engineering. It extends the scope and results of classical mathematical analysis by applying fuzzy logic to conventional mathematical objects, such as functions, sequences and series etc.

As a generalization of ordinary convergence for sequences of real numbers, the notion of statistical convergence was first introduced by Fast [9]. After then it was studied by many researchers like Salat [24], Fridy [10], Connor [3], Maddox [15], Fridy and Orhan [11], Nuray [20], Subrahmanyam [28], Kwon [14], Savas [25], Demirci [4], Tripathy [29], Savas and Mursaleen [27]. Different classes of statistically convergent sequences were introduced and investigated by Tripathy and Sen [34], Tripathy [29], Tripathy and Dutta [31], Tripathy and Sarma [33] etc. Móricz [16] extended statistical convergence from single to multiple real sequences. Nuray and Savas [21] first defined the concepts of statistical convergence and statistically Cauchy for sequences of fuzzy numbers. Some more works on statistical convergence are found in Altin [2], Esi and Ozdemir [8], Gokhan *et al.* [12], Tripathy and Baruah [29] etc.

Agnew [1] studied the summability theory of multiple sequences and obtained certain theorems which have already been proved for double sequences by the author himself. The different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner *et al.* [22], Sahiner and Tripathy [23]. Recently Savas and Esi [26] have introduced statistical convergence of triple sequences on probabilistic normed space. Later on, Esi [7] have introduced statistical convergence of triple sequences in topological groups. Some more recent works on statistical triple sequences are found on Kumar *et al.* [13], Esi [6], Dutta *et al.* [5] etc.

A fuzzy real number on R is a mapping $X : R \to L(=[0,1])$ associating each real number $t \in R$ with

its grade of membership X (t).Every real number r can be expressed as a fuzzy real number $\bar{r}$ as follows:

$$\bar{r}\,(t) = \begin{cases} 1 & if \quad t = r \\ 0 & otherwise \end{cases}$$

The α-level set of a fuzzy real number X , $0 < \alpha \le 1,$ denoted by $[X]^{\alpha}$ is defined as

$$[X]^{\alpha} = \{t \in R : X(t) \ge \alpha\}.$$

A fuzzy real number X is called convex if $X(t) \ge X(s) \wedge X(r) = \min(X(s), X(r)),$ where $s < t < r.$ If there exists $t_0 \in R$ such that $X(t_0) = 1,$ then the fuzzy real number X is called normal. A fuzzy real number X is said to be upper semi-continuous if for each $\varepsilon > 0, \backslash X^{-1}[0, a+\varepsilon)),$ for all $a \in L$ is open in the usual topology of $R.$ The set of all upper semi continuous, normal, convex fuzzy number is denoted by $R(L).$ The additive identity and multiplicative identity in $R(L)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.
Let D be the set of all closed bounded intervals $X = \left[X^L, X^R\right]$ on the real line R. Then

$X \le Y$ if and only if $X^L \le Y^L$ and $X^R \le Y^R.$ Also let $d(X,Y) = \max(|X^L - X^R|, |Y^L - Y^R|).$

Then (D, d) is a complete metric space.

Let $\bar{d} : R(L) \times R(L) \to R$ be defined by $\bar{d}(X,Y) = \sup_{0 \le \alpha \le 1} d([X]^{\alpha}, [Y]^{\alpha})$, for $X, Y \in R(L).$

Then $\bar{d}$ defines a metric on $R(L).$
Throughout the paper, $\bar{d}$ denote a metric.

2. Preliminaries and background

Throughout the paper c, c_0, ℓ_{∞} denote the spaces of convergent, null and bounded sequences respectively.
A triple sequence can be defined as a function $x : N \times N \times N \to R(C),$ where N, R and C denote the sets of natural numbers, real numbers and complex numbers respectively.

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is a triple infinite array of fuzzy real numbers X_{nlk} for all $n, k, l \in N$ and is denoted by $\langle X_{nkl} \rangle$ where $X_{nkl} \in R(L).$

The notion of statistical convergence for triplesequences depends on the density of the subsets of $N \times N \times N$.
The notion of the asymptoticdensity for subsets of $N \times N \times N$ as follows:

A subset E of $N \times N \times N$ is said to have density or asymptotic density $\rho(E),$ if

$\rho(E) = \lim_{p,q,r \to \infty} \sum_{n=1}^{p} \sum_{k=1}^{q} \sum_{l=1}^{r} \chi_E(n,k,l)$ exists, where χ_E is the characteristic function of E.

Obviously $\rho(E^c) = \rho(N \times N \times N - E) = 1 - \rho(E).$

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be statistically convergent in Pringsheim's sense to the fuzzy real number $X_0,$ if for all $\varepsilon > 0,$ the set $\rho\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl}, X_0) \ge \varepsilon\} = 0.$
We write $stat_3 - \lim_{n,k,l \to \infty} X_{nkl} = X_0.$

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be statistically null if it is statistically convergent to zero.

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be statistically bounded if there exists a real number μ such that $\rho\{(n,k,l) \in N \times N \times N : \bar{d},(X_{nkl}, \bar{0}) > \mu\} = 0.$.

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be statistically Cauchy if for every $\varepsilon > 0$, there exists $p = p(\varepsilon)$, $q = q(\varepsilon)$ and $r = r(\varepsilon) \in N$ such that
$\rho(\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl}, X_{pqr}) \geq \varepsilon\}) = 0.$

A fuzzy real-valued triple sequence $X = \langle X_{nkl} \rangle$ is said to be statistically regularly convergent if it convergent in Pringsheim's sense and in addition the following statistical limits holds:

$$stat_3 - \lim_{n \to \infty} X_{nkl} = L_{kl}(k,l \in N);$$

$$stat_3 - \lim_{k \to \infty} X_{nkl} = L_{nl}(n,l \in N);$$

and

$$stat_3 - \lim_{l \to \infty} X_{nkl} = L_{nl}(n,k \in N);$$

Different classes of fuzzy sequencespaces were introduced and investigated by many authors.Recentworks on fuzzy triple sequences are found in Nath and Roy ([17], [18], [19]).

A fuzzy real-valued triple sequence $\langle X_{nkl} \rangle$ is said to be Cesáro summable to a fuzzy real number X_0, if

$$\lim_{u,v,w \to \infty} \bar{d}\left(\frac{1}{uvw}\left(\sum_{n=1}^{u}\sum_{l=1}^{v}\sum_{k=1}^{w} X_{nkl}\right), X_0 \right) = 0.$$

A fuzzy real-valued triple sequence $\langle X_{nkl} \rangle$ is said to be strongly *p*-Cesáro summable to a fuzzy real number X_0, if $\lim_{u,v,w \to \infty} \frac{1}{uvw} \sum_{n=1}^{u}\sum_{l=1}^{v}\sum_{k=1}^{w} \left(\bar{d}(X_{nkl}, X_0)\right)^p = 0.$

We denote the space of all strongly *p*-Cesáro summable triple sequences of fuzzy real numbers by $(w^F)_3^P$.

Let $\langle X_{nkl} \rangle$ and $\langle Y_{nkl} \rangle$ be two triple sequences of fuzzy real numbers, then $X_{nkl} = Y_{nkl}$ for almost all *n*, *l* and *k* (in short *a.a. n*, *k* and *l*) if $\rho(\{(n,k,l) \in N \times N \times N : X_{nkl} \neq, Y_{nkl}\}) = 0.$

Remark 2.1. *If a sequence space E^F is solid, then it is monotone.*

Throughout the article $(w^F)_3$, $(\ell_\infty^F)_3$, $(\bar{c}^F)_3^R$, $(\bar{c}_0^F)_3^R$, $(\bar{c}^F)_3^P$, $(\bar{c}_0^F)_3^P$ denote the spaces of all, bounded, statistically regularly convergent, statistically regularly null, statisticallyconvergent in Pringsheim's sense and statistically null in Pringsheim's sense triple sequence spaces of fuzzy real numbers respectively.

Also the following sequence spaces are introduced:

$(\overline{c}^{F})_3^{RB} = (\overline{c}^{F})_3^{R} \cap (\ell_\infty^F)_3;$

$(\overline{c}_0^{F})_3^{RB} = (\overline{c}_0^{F})_3^{R} \cap (\ell_\infty^F)_3;$

$(\overline{c}^{F})_3^{PB} = (\overline{c}^{F})_3^{P} \cap (\ell_\infty^F)_3;$

$(\overline{c}_0^{F})_3^{PB} = (\overline{c}_0^{F})_3^{P} \cap (\ell_\infty^F)_3.$

From the above definitions, it follows that $(c)_3^P \subset (\overline{c}^F)_3^P, (c_0)_3^P \subset (\overline{c}_0^F)_3^P, (c)_3^{PB} \subset (\overline{c}^F)_3^{PB}$ and $(c_0)_3^{PB} \subset (\overline{c}_0^F)_3^{PB}$ and the inclusions are proper.

3. Main Results:

Theorem 3.1. Every convergent triple sequence of fuzzy real number is statistically convergent, but the converse is not necessarily true.

Proof. Since the asymptotic density of finite subsets of $N \times N \times N$ is zero, it follows that every convergent triple sequence of fuzzy real number is statistically convergent.

But the converse is not necessarily true as seen from the following example.

Example 3.1. For every $t \in R,$ consider a sequence $X = \langle X_{nkl} \rangle$ of fuzzy numbers defined as follows:

For $n = p^2, k = q^2, l = r^2,\ p, q, r \in N$

$$X_{nkl}(t) = \begin{cases} 0, & if \quad t \in (-\infty, nkl-1) \cup (nkl+1, \infty) \\ t-(nkl-1), & if \quad t \in [nkl-1, nkl] \\ -t+(nkl+1), & if \quad t \in [nkl, nkl+1] \end{cases}$$

Otherwise $X_{nlk} = X_0,$ where X_0 is given by

$$X_0(t) = \begin{cases} 0, & if \quad t \in (-\infty, 0) \cup (2, \infty) \\ t, & if \quad t \in [0,1] \\ -t+2, & if \quad t \in [1,2] \end{cases}$$

Now, for $0 < \varepsilon < 1,$ we have

$$\{(n,k,l) \in N \times N \times N : \overline{d}(X_{nkl}, X_0) \geq \varepsilon\} \subset \{(n,k,l) \in N \times N \times N : n = p^2, k = q^2, l = r^2, p, q, r \in N\}$$

Since, the later set has triple density zero, it follows that

$$\rho(\{(n,k,l) \in N \times N \times N : d(X_{nkl}, X_0) \geq \varepsilon\}) = 0.$$

$\therefore stat_3 - \lim_{n,k,l \to \infty} X_{nkl} = X_0.$ But, the sequence $X = \langle X_{nkl} \rangle$ is not ordinarily convergent to X_0. ■

With the usual technique, the following theorem can be proved easily.

Theorem 3.2. *If a triple sequence $\langle X_{nkl} \rangle$ of fuzzy real numbers is statistically convergent to L, then L is determined uniquely.*

We now give algebraic characterization of statistical limit for triple sequences of fuzzy real numbers in the following theorem.

Theorem 3.3. *Let $\langle X_{nkl} \rangle$ and $\langle Y_{nkl} \rangle$ be two triple sequences of fuzzy real numbers. Then*

(i) If $\langle X_{nkl} \rangle$ is statistically convergent to X_0 and $c \in R$, then $\langle cX_{nkl} \rangle$ is statistically convergent to cX_0.

(ii) If $\langle X_{nkl} \rangle$ and $\langle Y_{nkl} \rangle$ are statistically convergent to fuzzy numbers X_0 and Y_0 respectively, then $\langle X_{nkl} + Y_{nkl} \rangle$ is statistically convergent to $X_0 + Y_0$.

Proof. (i) Let $stat_3 - \lim_{n,k,l\to\infty} X_{nkl} = X_0$.

Then for any $\varepsilon > 0$,

$$\therefore \rho\left(\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl}, X_0) \geq \varepsilon\}\right) = 0.$$

Let $\alpha \in [0,1]$. Let X_{nkl}^{α} and X_0^{α} be α-level sets of X_{nkl} and X_0 respectively.

Since $d(cX_{nkl}^{\alpha}, cX_0^{\alpha}) = |c| d(X_{nkl}^{\alpha}, X_0^{\alpha})$, for $c \in R$.

$$\Rightarrow \sup_{\alpha} d(cX_{nkl}^{\alpha}, cX_0^{\alpha}) = |c| \sup_{\alpha} d(X_{nkl}^{\alpha}, X_0^{\alpha})$$

$$\Rightarrow \bar{d}(cX_{nkl}, cX_0) = |c| \bar{d}(X_{nkl}, X_0)$$

This gives for $\varepsilon > 0$,

$$\frac{1}{pqr}\left|\{(n,k,l) \in N \times N \times N : n \leq p; k \leq q, l \leq r : \bar{d}(cX_{nkl}, cX_0) \geq \varepsilon\}\right|$$

$$\leq \frac{1}{pqr}\left|\left\{(n,k,l) \in N \times N \times N : n \leq p; k \leq q, l \leq r : \bar{d}(X_{nkl}, X_0) \geq \frac{\varepsilon}{|c|}\right\}\right|.$$

$$\lim_{p,q,r\to\infty} \frac{1}{pqr}\left|\{(n,k,l) \in N \times N \times N : n \leq p; k \leq q, l \leq r : \bar{d}(cX_{nkl}, cX_0) \geq \varepsilon\}\right| = 0,$$

$$\therefore \rho\{(n,k,l) \in N \times N \times N : \bar{d}(cX_{nkl}, cX_0) \geq \varepsilon\} = 0.$$

Hence $stat_3 - \lim_{n,k,l\to\infty} cX_{nkl} = cX_0$.

(ii) Let $stat_3 - \lim_{n,k,l\to\infty} cX_{nkl} = cX_0$ and $stat_3 - \lim_{n,k,l\to\infty} cY_{nkl} = cY_0$.

$$\therefore \rho\left(\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl}, X_0) \geq \varepsilon\}\right) = 0 \text{ and } \rho\left(\{(n,k,l) \in N \times N \times N : \bar{d}(Y_{nkl}, Y_0) \geq \varepsilon\}\right) = 0.$$

Let $\alpha \in [0,1]$. Let $X_{nkl}^{\alpha}, Y_{nkl}^{\alpha}, X_0^{\alpha}$ and Y_0^{α} be the α-level sets of X_{nkl}, Y_{nkl}, X_0 and Y_0 respectively.

Now $d(X_{nkl}^{\alpha} + Y_{nkl}^{\alpha}, X_0^{\alpha} + Y_0^{\alpha}) \leq d(X_{nkl}^{\alpha}, X_0^{\alpha}) + d(Y_{nkl}^{\alpha}, Y_0^{\alpha})$

$$\Rightarrow \sup_{\alpha} d(X_{nkl}^{\alpha} + Y_{nkl}^{\alpha}, X_0^{\alpha} + Y_0^{\alpha}) \leq \sup_{\alpha} d(X_{nkl}^{\alpha}, X_0^{\alpha}) + d(Y_{nkl}^{\alpha}, Y_0^{\alpha})$$

$$\Rightarrow \bar{d}\ (X_{nkl}^{\alpha} + Y_{nkl}^{\alpha}, X_0^{\alpha} + Y_0^{\alpha}) \leq \bar{d}(X_{nkl}^{\alpha}, X_0^{\alpha}) + \bar{d}(Y_{nkl}^{\alpha}, Y_0^{\alpha})$$

For a given $\varepsilon > 0$,

$$\frac{1}{pqr}\left|\{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r : \bar{d}(X_{nkl} + Y_{nkl}, X_0 + Y_0) \ge \varepsilon\}\right|$$

$$\le \frac{1}{pqr}\left|\{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r : \bar{d}(X_{nkl}, X_0) + \bar{d}(Y_{nkl}, Y_0)\} \ge \varepsilon\right|$$

$$\le \frac{1}{pqr}\left|\left\{n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r : \bar{d}(X_{nkl}, X_0) \ge \frac{\varepsilon}{2}\right\}\right| +$$

$$\frac{1}{pqr}\left|\left\{n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r : \bar{d}(Y_{nkl}, Y_0) \ge \frac{\varepsilon}{2}\right\}\right|.$$

$$\lim_{p,q,r\to\infty} \frac{1}{pqr}\left|\{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r : \bar{d}(X_{nkl} + Y_{nkl}, X_0 + Y_0) \ge \varepsilon\}\right| = 0,$$

$$\therefore \rho\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl} + Y_{nkl}, X_0 + Y_0) \ge \varepsilon\} = 0.$$

Hence $stat_3 - \lim_{n,k,l\to\infty} X_{nkl} + X_{nkl} = X_0 + Y_0$. ■

The following result, known as the **decomposition theorem** for statistically convergent fuzzy real-valued triple sequence space.

Theorem 3.4. *The following statements are equivalent*:

(i) *The triple sequence* $\langle X_{nkl} \rangle$ *of fuzzy real numbers is statistically convergent to* X_0.

(ii) *There exists a triple sequence* $\langle Y_{nkl} \rangle \in (\bar{c}^F)_3^P$ *such that* $X_{nkl} = Y_{nkl}$ *for a.a. n, k and l.*

(iii)There exists a subset $M = \{n_p, k_q, l_r) \in N \times N \times N : p, q, r \in N\}$ *of* $N \times N \times N$ *such that* $\rho(M) = 1$ *and* $\langle X_{n_p k_q l_r} \rangle \in (\bar{c}^F)_3^P$.

(iv)*There exists two triple sequences* $\langle A_{nkl} \rangle$ *and* $\langle B_{nkl} \rangle$ *of fuzzy real numbers such that* $X_{nkl} = A_{nkl} + B_{nkl}$ *for all* $n,k,l \in N$, *where* $\langle A_{nkl} \rangle$ *converges to* X_0 *and* $\langle B_{nkl} \rangle \in (\bar{c}_0^F)_3^P$.

Proof. **(*i*) ⇒ (*ii*).** Let $stat_3 - \lim_{n,k,l\to\infty} X_{nkl} = X_0$. Then for every $\varepsilon > 0$,

$$\rho(\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl}, X_0) \ge \varepsilon\}) = 0.$$

Now selecting the increasing sequences (T_j), (U_j) and (V_j) of natural numbers such that if

$p > T_j, q > U_j$ and $r > V_j$, then $\frac{1}{pqr}\left\{(n,k,l) \in N \times N \times N : n \le p; k \le q; l \le r : \bar{d}(X_{nkl}, X_0) \ge \frac{1}{j}\right\} < \frac{1}{j}$, where $|E|$ denote the cardinality of the set E.

Consider the sequence $\langle Y_{nkl} \rangle$ defined as follows:

$Y_{nkl} = X_{nkl}$, if $n \le T_1$ or $k \le U_1$ or $l \le V_1$.

Next for all (n,k,l) with $T_j < n \le T_{j+1}$ or $U_j < k \le U_{j+1}$, or $V_j < l \le V_{j+1}$,

let $Y_{nkl} = X_{nkl}$, if $\bar{d}(X_{nkl}, X_0) < \frac{1}{j}$, otherwise, $Y_{nkl} = X_0$.

Let $\varepsilon > 0$ be given and choosing j such that $\varepsilon < \frac{1}{j}$.

For $n > T_j, k > U_j$ and $l > V_j$, we find that $\bar{d}(Y_{nkl}, X_0) < \varepsilon$.

The fact that $Y_{nkl} = X_{nkl}$, for *a.a. n, k and l.* follows from the following inclusion:

Let $T_j < n \le T_{j+1}, U_j < k \le U_{j+1}$ and $V_j < l \le V_{j+1}$, then

$$\{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r \; and \quad X_{nkl} \ne Y_{nkl}\}$$

$$\subseteq \{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r \; and \quad \bar{d}(X_{nkl}, X_0) \ge \frac{1}{j}\}$$

$$\Rightarrow \frac{1}{pqr} \left| \{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r \; and \quad X_{nkl} \ne Y_{nlk}\} \right|$$

$$\le \frac{1}{pqr} \left| \left\{ (n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r \; and \quad \bar{d}(X_{nkl}, X_0) \ge \frac{1}{j} \right\} \right| < \frac{1}{j},$$

$$\Rightarrow \lim_{p,q,r \to \infty} \frac{1}{pqr} \left| \{(n,k,l) \in N \times N \times N : n \le p; k \le q, l \le r \; and \quad X_{nkl} \ne Y_{nkl}\} \right| = 0.$$

(ii)* ⇒ *(iii) Let there exists a sequence $\langle Y_{nkl} \rangle \in (\bar{c}^F)_3^P$ such that $X_{nkl} = Y_{nkl}$ for *a.a. n, k* and *l.*

Let $M = \{n,k,l) \in N \times N \times N : X_{nkl} = Y_{nkl}\}$, then $\rho(M) = 1$.

Now enumerate *M* as $M = \{n_p, k_q, l_r) \in N \times N \times N : p,q,r \in N\}$ on neglecting the rows and columns those contain finite number of elements.

Clearly $\rho(M) = 1$ and $\bar{d}(X_{n_p k_q l_r}, X_0) = \bar{d}(Y_{n_p k_q l_r}, X_0) \to 0$ as $p,q,r \to \infty$.

$$\Rightarrow \left\langle X_{n_p k_q l_r} \right\rangle \in (\bar{c}^F)_3^P.$$

(iii)⇒ (iv). Let $\langle X_{nkl} \rangle$ be a triple sequence and there exists $M = \{n_p, k_q, l_r) \in N \times N \times N : p,q,r \in N\}$ be such that $\rho(M) = 1$ and $\lim_{p,q,r \to \infty} X_{n_p k_q l_r} = X_0$.

Now, two triple sequences $\langle A_{nkl} \rangle$ and $\langle B_{nkl} \rangle$ of fuzzy real numbers are constructed as follows:

$$A_{nkl} \begin{cases} X_{nkl}, & if \quad (n,k,l) \in N, \\ X_0, & otherwise \end{cases}$$

$$B_{nkl} = \begin{cases} 0, & if \quad (n,k,l) \in N, \\ X_{nkl} - X_0, & otherwise \end{cases}$$

From the above construction, it is clear that $\langle A_{nkl} \rangle$ converges to X_0 and $\langle B_{nkl} \rangle \in (\bar{c}_0^F)_3^P$ and

$X_{nkl} = A_{nkl} + B_{nkl}$ for all $n,k,l \in N$,

(iv) ⇒ (i). Let there exists two triple sequences $\langle A_{nkl} \rangle$ and $\langle B_{nkl} \rangle$ of fuzzy real numbers such that

$X_{nkl} = A_{nkl} + B_{nkl}$ for all $n,k,l \in N$, where $\langle A_{nkl} \rangle$ converges to X_0 and $\langle B_{nkl} \rangle \in (\bar{c}_0^F)_3^P$.

For any $\varepsilon > 0$,

Let $A = \left\{(n,k,l) : \bar{d}(X_{nkl}, X_0) < \frac{\varepsilon}{2}\right\}$ and $B = \left\{(n,k,l) : \bar{d}(X_{nkl}, \bar{0}) < \frac{\varepsilon}{2}\right\}$.

Then clearly

$\rho(A) = \rho(B) = 1.$ Let $E = A \cap B,$ then $\rho(E) = 1$ and

$\therefore \rho\left(\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl}, X_0) < \varepsilon\}\right) \geq \rho(E) = 1.$

$\Rightarrow \rho\left(\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl}, X_0) \geq \varepsilon\}\right) = 0.$

$\therefore stat_3 - \lim_{n,k,l \to \infty} X_{nkl} = X_0.$ ■

Theorem 3.5. *A triple sequence $\langle X_{nkl} \rangle$ of fuzzy real numbers is statistically convergent, if and only if $\langle X_{nkl} \rangle$ is a statistical Cauchy sequence.*

Proof. Let $stat_3 - \lim_{n,k,l \to \infty} X_{nkl} = X_0$.

Then for each $\varepsilon \geq 0$,

$\rho\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl}, X_0) \geq \varepsilon\} = 0.$

Choose numbers *p, q* and *r* such that $\bar{d}(X_{pqr}, X_0) \geq \varepsilon.$

Now let

$A = \{(n,k,l) \in N \times N \times N, n \leq u, k \leq v, l \leq w : \bar{d}(X_{nkl}, X_{pqr}) \geq \varepsilon\};$

$B = \{(n,k,l) \in N \times N \times N, n \leq u, k \leq v, l \leq w : \bar{d}(X_{nkl}, X_0) \geq \varepsilon\};$

$C = \{(p,q,r) \in N \times N \times N : \bar{d}(X_{pqr}, X_0) \geq \varepsilon\}.$

Then $A \subseteq B \cup C$ and therefore $\rho(A) \leq \rho(B) + \rho(C) = 0.$ Hence $\langle X_{nkl} \rangle$ is statisticallyCauchy.

Conversely, let $\langle X_{nkl} \rangle$ be statistically Cauchy. Then for a given $\varepsilon > 0$, there exists $n_0(\varepsilon), k_0(\varepsilon)$ and $l_0(\varepsilon) \in N$ such that $\rho(\{(n,k,l) \in N \times N \times N : \bar{d}(X_{nkl}, X_{pqr}) \geq \varepsilon\}) = 0.$

i.e. $\bar{d}(X_{nkl}, X_{n_0k_0l_0}) < \varepsilon,$ for *a.a. n, k and l.* Now choose n_1, k_1, l_1 such that $X_{nkl} \in U_1(X_{n_1k_1l_1}, 1),$ for *a.a. n, k and l.* Next choose n_2, k_2, l_2 such that $X_{nlk} \in U_2(X_{n_2l_2k_2}, 2^{-1}),$ for *a.a. n, k and l.*

Let $U^1 = U_1 \cap U_2$. Clearly U^1 contains X_{nlk} for *a.a.* for *a.a. n, k and l* and the diameter of U^1 is less than or equal to 1. Similarly one can get a closed ball $U_3(X_{n_3k_3l_3}, 2^{-2})$, which contains X_{nlk} for *a.a. n, k and l.* The diameter of U^2 is less than or equal to 2^{-1}. Proceeding in this way, a nest of closed balls $\{U^j\}$, is obtained such that $X_{nkl} \in U^j$, for *a.a. n, k and l* and the diameter of U^j is less than or equal to 2^{1-j}. Then by the theorem of nested property of closed fuzzy sets, $\bigcap_{j=1}^{\infty} U^j$ contains exactly one element.

Let $\bigcap_{j=1}^{\infty} U^j = \{X_0\}$. Let $\varepsilon > 0$ be given and let *t* be a constant satisfying $\varepsilon > 2^{1-t}$. Then $X_{nlk} \in U^t$, for *a.a. n, k and l,* which gives $\bar{d}(X_{nkl}, X_0) \le 2^{1-t} < \varepsilon$, for *a.a. n, k and l.*

This implies that $stat_3 - \lim_{n,k,l \to \infty} X_{nkl} = X_0$. ■

Theorem 3.6. (*i*) *Let* $p \in (0, \infty)$. *If a triple sequence* $\langle X_{nkl} \rangle$ *of fuzzy real number is strongly p-Cesáro summable to a fuzzy number* X_0, *then it is also statistically convergent to* X_0.

(*ii*) *Let* $p \in (0, \infty)$. *If a bounded triple sequence* $\langle X_{nkl} \rangle$ *of fuzzy real number is statistically convergent to* X_0, *then it is strongly p-Cesáro summable to* X_0.

Proof. (*i*) Let $\langle X_{nkl} \rangle \in (w^F)_3^p$ be such that $\langle X_{nkl} \rangle$ is strongly *p*-Cesáro summable to a fuzzy number X_0. Then $\lim_{u,v,w \to \infty} \frac{1}{uvw} \sum_{n=1}^{u} \sum_{l=1}^{v} \sum_{k=1}^{w} \left(\bar{d}(X_{nkl}, X_0) \right)^p = 0.$

Now for any $\varepsilon > 0$,

$$\frac{1}{uvw} \sum_{n=1}^{u} \sum_{l=1}^{v} \sum_{k=1}^{w} \left(\bar{d}(X_{nkl}, X_0) \right)^p \ge \frac{1}{uvw} \left| \{(n,k,l) \in N \times N \times N : n \le u; k \le v, l \le w : \left(\bar{d}(X_{nkl}, X_0) \right)^p \ge \varepsilon\} \right|.$$

Now taking limits as $u, v, w \to \infty$,

$$stat_3 - \lim_{n,k,l \to \infty} X_{nkl} = X_0.$$

Again for a bounded triple sequence $\langle X_{nkl} \rangle$, let $stat_3 - \lim_{n,k,l \to \infty} X_{nkl} = X_0$.

Let $K = \bar{d}(X_{nkl}, \bar{0}) + \bar{d}(X_0, \bar{0})$.

Let $\varepsilon > 0$ be given, then there exist u_0, v_0, w_0 such that

$$\frac{1}{uvw} \left| \left\{ (n,k,l) \in N \times N \times N : n \le u; k \le v, l \le w; \bar{d}(X_{nkl}, X_0) \ge \left(\frac{\varepsilon}{2} \right)^{\frac{1}{p}} \right\} \right| < \frac{\varepsilon}{2K^p},$$ **for all** $u > u_0, v > v_0$ **and** $w > w_0$.

Let $L_{uvw} = \left\{ (n,k,l) \in N \times N \times N : n \le u; k \le v; l \le w; \left(\bar{d}(X_{nkl}, X_0) \right)^p \ge \frac{\varepsilon}{2} \right\}$.

Now for all $u > u_0, v > v_0$ and $w > w_0$,

$$\frac{1}{uvw}\sum_{n=1}^{u}\sum_{k=1}^{v}\sum_{l=1}^{w}\left(\bar{d}(X_{nkl}, X_0)\right)^p = \frac{1}{uvw}\left\{\sum_{(n,k,l)\in L_{uvw}}\sum\sum\left(\bar{d}(X_{nkl}, X_0)\right)^p\right\} + \frac{1}{uvw}\left\{\sum_{(n,k,l)\notin L_{uvw}}\sum\sum\left(\bar{d}(X_{nkl}, X_0)\right)^p\right\}$$

$$< \frac{1}{uvw}\left\{uvw\frac{\varepsilon}{2K^p}.K^p + uvw\frac{\varepsilon}{2}\right\} < \varepsilon.$$

$$\therefore \lim_{u,v,w\to\infty}\frac{1}{uvw}\sum_{n=1}^{u}\sum_{k=1}^{v}\sum_{l=1}^{w}\left(\bar{d}(X_{nkl}, X_0)\right)^p = 0.$$

Hence $\langle X_{nkl}\rangle$ is strongly p-Cesáro summable to X_0. ■

In view of the above theorem , the following results can be obtained.

Corollary 3.1. *Let* $0 < p < q < \infty$. *Then* $(w^F)_3^q \subset (w^F)_3^P$ *and* $(w^F)_3^P \cap (\ell^F)_3^\infty = (w^F)_3^q \cap (\ell^F)_3^\infty$.

Corollary 3.2. *If a bounded triple sequence* $\langle X_{nkl}\rangle$ *of fuzzy real number is statistically convergent to* X_0, *then it is Cesáro summable to* X_0.

Remark 3.1. *If a bounded triple sequence* $\langle X_{nkl}\rangle$ *of fuzzy real number is Cesáro summable, then* $\langle X_{nkl}\rangle$ *may not be statistically convergent*.

It follows from the following example.

Example 3.1. Let $X = \langle X_{nkl}\rangle$ be defined as $X_{nkl} = (-1)^n$, for all *k*, *l*. Then *X* is Cesáro summable but not statistically convergent.

Remark 3.2. *If* $X = \langle X_{nkl}\rangle$ *is unbounded triple sequence of fuzzy real number, thenX is statistically convergent but X is not necessarily strongly p-Cesáro summable.*

The remark follows from the following example.

Example 3.2. Let $X = \langle X_{nkl}\rangle \in R(L)$ be defined as

For $n = i^2, i \in N$ and for all $k, l \in N$,

$$X_{nkl}(t) = \begin{cases} 1+\frac{t}{\sqrt{n}}, & for \quad -\sqrt{n} \le t \le 0 \\ 1-\frac{t}{\sqrt{n}}, & for \quad 0 \le \mathrm{t} \le \sqrt{n} \\ 0, & otherwise. \end{cases}$$

For $n \ne i^2, i \in N$ and for all $k, l \in N$, $X_{nkl}(t) = \bar{0}$.

The sequence *X* is unbounded and statistically convergent to $\bar{0}$.

But for p = 1, $\lim_{u,v,w\to\infty}\frac{1}{uvw}\sum_{n=1}^{u}\sum_{k=1}^{v}\sum_{l=1}^{w}\left(\bar{d}(X_{nkl}, \bar{0}\right)^p = 1$

Hence X is not strongly Cesáro summable to $\bar{0}$. ■

Conclusion:

Convergence theory is used as a basic tool in, measure spaces, sequences of random variables, information theory etc. We have introduced and studied some new classes of statistical convergence of sequences of fuzzy real numbers having multiplicity greater than two. Certain Theorems regarding uniqueness of limit, algebraic characterization of statistical limit for triple sequences of fuzzy numbers are obtained. The decomposition theorem is proved. The inclusion relations are derived. The fuzzy real-valued Cesáro summable triple sequence space is also introduced. A relation between strongly p-Cesáro summability and bounded statistically convergent triple sequences is established.

References

[1] R. P. Agnew, On summability of multiple sequences, *American Journal of Mathematics*; **1(4)**, 62-68, (1934).

[2] Y. Altin, A note on lacunary statistically convergent double sequences of fuzzy numbers,*Commun. Korean Math. Soc.*; **23(2)**, 179–185, (2008).

[3] J. S. Connor, The statistical and strong *p*-Cesáro convergence of sequences; *Analysis*; **8**, 47-63, (1988).

[4] K. Demirci, On lacunary statistical limit points, *DemonstratoMathematica*; **35(1)**, 93-101, (2002).

[5] A. J. Dutta, A. Esi, B. C. Tripathy, Statistically convergence triple sequence spaces defined by Orlicz function, *Journal of Mathematical Analysis*; **4(2)**, 16-22, (2013).

[6] A. Esi, λ_3 -Statistical convergence of triple sequences on probabilistic normed space, *Global Journal of Mathematical Analysis*; **1(2)**, 29-36, (2013).

[7] A. Esi, Statistical convergence of triple sequences in topological groups, Annals of theUniversity of Craiova, *Mathematics and Computer Science Series*; **40(1)**, 29-33, (2013).

[8] A. Esi, M. K. Ozdemir, Generalized m-statistical convergence in probabilistic normed space, *J. Comput. Anal. Appl.*; **13(5)**, 923-932, (2011).

[9] H. Fast, Surla convergence statistique, *Colloq. Math*.; **2**, 241-244, (1951).

[10] J. A. Fridy, On statistical convergence, *Analysis*; **5**, 301-313, (1985).

[11] J. A. Fridy, C. Orhan, Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc.*; **125(12)**, 3625-3631, (1997).

[12]. A. Gokhan, M. Et, M. Mursaleen, Almost lacunary statistical and strongly almostlacunary Convergenceof sequences of fuzzy numbers, *Math. Comput. Modelling*, **49(3-4),** 548–555, (2009).

[13]. Kumar, V. Kumar, S. S. Bhatia, Multiple sequence of Fuzzy numbers and their statistical convergence, *Mathematical Sciences, Springer*; **6(2)**, 1-7, (2012).

[14] J. S. Kwon, On statistical and *p*-Cesáro convergence of fuzzy numbers, *Korean J. Comput. Appl. Math*.; **7**, 195–203, (2000).

[15] I. J. Maddox, A tauberian condition for statistical convergence, *Math. Proc. Camb. PhilSoc*.; **106**, 272-280, (1989).

[16] F. Moričz, Statistical convergence of multiple sequences, *Arch. Math*.; **81**, 82–89, (2003).

[17] M. Nath, S. Roy, On fuzzy real-valued multiple sequence spaces $_3\ell^F(p)$, *International Journal of Emerging Trends in Electrical and Electronics*; **11(4)**, 103-107, (2015).

[18] M. Nath, S. Roy, Some new classes of fuzzy real-valued ideal convergent multiple sequence spaces,*Asian Journal of Mathematics and Computer Research*; **11(4)**, 272-288, (2016).

[19] M. Nath, S. Roy, Some new classes of ideal convergent difference multiple sequences of fuzzy real numbers, *Journal of Intelligent and Fuzzy systems*; **31(3)**, 1579-1584, (2016).

[20] F. Nuray, Lacunary statistical convergence of sequences of fuzzy numbers, *Fuzzy Sets Systems*; **99**, 353-356, (1998).

[21] F, Nuray, E. Savas, Statistical convergence of sequences of fuzzy numbers. *Math. Slovaca*; **45**, 269–273, (1995).

[22] A. Şahiner, M. Gürdal, F. K. Düden**,** Triple sequences and their statistical convergence, *Seluk J. Appl. Math.*; **8(2)**, 49-55, (2007).

[23] A. Sahiner, B. C. Tripathy, Some *I* -related Properties of Triple Sequences, *Selcuk J. Appl. Math.*; **9(2)**, 9-18, (2008).

[24] T. Šalát, On statistically convergent sequences of real numbers, *Math. Slovaca*; **30**, 139-150, (1980).

[25] E.Savas, On statistically convergent sequences of fuzzy numbers. *Inform. Sci.*; **137(1-4)**, 277-282, (2001).

[26] E. Savas, A. Esi, Statistical convergence of triple sequences on probabilistic normed space, Annals of the University of Craiova, *Mathematics and Computer Science Series*; **39(2)**, 226 -236, (2012).

[27] E. Savas, M. Mursaleen, On statistically convergent double sequences of fuzzy numbers, *Inform. Sci.*; **162(3-4)**, 183-192, (2004).

[28] P. V. Subrahmanyam, Cesáro summability of fuzzy real numbers, *J. Analysis*; **7**, 159-168, (1999).

[29] B. C. Tripathy, Statistically convergent double sequences, *Tamkang J. Math.*; **34(3)**, 231-237, (2003).

[30] B. C. Tripathy, A. Baruah, Lacunary statistically convergent and lacunary strongly convergent generalized difference sequences of fuzzy real numbers, *Kyungpook Math. J.*, **50**, 565-574. (2010).

[31] B. C. Tripathy, A. J. Dutta, Statistically convergence and Cesáro summable double sequences of fuzzy real numbers, *Soochow journal of Mathematics*; **33(4),** 835-848, (2007).

[32] B. C. Tripathy, A. J. Dutta, Statistically convergence triple sequence spaces defined by Orlicz function, *Journal of Mathematical Analysis*; **4(2),** 16-22, (2013).

[33] B.C. Tripathy, B. Sarma, Statistically convergent difference double sequence spaces, *Acta Math. Sinica(Eng. Ser.*); **24(5)**, 737-742, (2008).

[34] B. C. Tripathy, M. Sen, On generalized statistically convergent sequences, *Indian Jour. Pure Appl. Math.*; **32(11)**, 1689-1694, (2001).

[35] L. A. Zadeh, Fuzzy sets, *Information and Control*; **8**, 338-353, (1965).