ISSN:2320-0790

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An International Journal of Advanced Computer Technology

THE FINITE ELEMENT METHOD FOR THE HIGH-ORDER DUAL-PHASE-LAGGING HEAT CONDUCTION MODEL

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Abstract: The non-Fourier heat conduction theory is appropriate for heat and mass transfer in micro-scale or nano-scale time and space condition. This article first expands the Dual-Phase-Lagging mathematical model to second-order term so as to describe the heat behaviour more accurately. Then dealing with the temperature variation about time and space by finite discretion, the coefficients of stiffness matrix are obtained. Initial and boundary conditions are defined according to practical environment.

Keywords: Heat Transfer, Fourier heat conduction

I. INTRODUCTION

Classical Fourier heat conduction law describes most conventional heat phenomenon, the equation of which is parabolic form. The drawback of Fourier model is the assumption of infinite heat propagation speed. In Practical cases, heat conduction propagates and diffuses in certain speed which depends on the properties of various mediums. Taken into consideration the deficiency, Cattaneo and Vernottee respectively put forward the hyperbolic heat conduction (HHC) equation to further explain the non-Fourier effect in some extreme conditions, such as laser irradiation, cryosurgery, micro-scale or nano-scale heat conduction [1][2]. The introduction of relaxation time τ reflects the finity of the heat propagation velocity as the wave form. However, there have been extensive researches on the analytical and numerical solutions to the hyperbolic heat conduction equation. As the influence of microstructure on the macro features become more increasingly important, the HHC equation is not sufficient to depict the transient process. Therefore, Tzou proposed the Dual-Phase-Lagging (DPL) heat conduction model, introducing both heat flux relaxation time τ_q and temperature gradient relaxation time τ_T to the Fourier heat conduction law as in equation (1-a) [3][4]:

$$q(t+\tau_q, r) = -k\nabla T(t+\tau_T, r)$$

(1-a)

q — heat flux intensity;
t — time;
τ_q — heat flux relaxation time;
r — position vector;
k—thermal conductivity;
T — temperature;
τ_T — temperature gradient relaxation time;

Coupled with local energy balance equation, the DPL heat conduction equation can be obtained, the highest order of which lies in the terms of Taylor expansion of the correctional Fourier heat conduction law about time t. The phase lag of temperature τ_T reflects the diffusion-like feature, while the phase lag of heat flux τ_q describes the wave-like feature [5].

As for the solution methods of DPL heat conduction equation, many scholars give some various ideas to obtain the temperature distribution and discuss the different influence of temperature phase lag τ_T and heat flux lag τ_q on the results. B. Wang obtained the DPL equation where the highest derivative order of temperature to time is three and investigated the relationship between thermal stress and normalized position, normalized time. Then the variation of thermal stress intensity factor with the normalized crack length, normalized time, the ratio of τ_T to τ_q and the crack growth for various values of the applied thermal stress intensity factor are analyzed. Furthermore, the thermal shock resistance is evaluated through stress-based failure criterion and toughness-based criterion [6]. Mauro Fabrizio indicated the restriction conditions under the second law of thermal dynamics for the two kinds of DPL heat conduction equations and then analyzed the asymptotic stability for memory kernel with exponential integrability property [7]. R. Quintanilla first considered the isotropic and homogeneous DPL partial differential equation of third order in time, then proved the existence of solution in Hilbert space with the help of semigroup theory and examined the exponential stability of solutions with time when the phase lag parameters satisfy certain conditions as well as the instability under the contrary conditions [8]. Mohsen Torabi first established DPL equation in cylinder coordinate, then respectively applied variables separation with Duhamel theorem and implicit difference method to get the analytical and numerical solution, which coincides with each other well. The results lead to the conclusion that the same value of τ_T and τ_q produces the same temperature variation trend as Fourier model and while τ_T is higher than τ_q the over-diffusion happens [5]. J.K. Chen proposed the DPL diffusion (DPLD) model with the delayed time of mass flux vector τ_j and the delayed time of density gradient τ_ρ and adopted Laplace transform to solve the DPLD equation, and derived three kinds solutions under cylinder or spherical or planar coordinate for fiber-reinforced and particulate metal matrix composites. Through compared with previous experiment results, the validity and accuracy of this method are confirmed [10]. J. Ghazanfarian adopted Laplace transformation and Riemann-sum approximation to solve 1-D DPL heat conduction equation which involves the effect of boundary phonon scattering and compared the numerical simulation results with both of Ballistic-Diffusive equation and Boltzmann equation under different Knudsen number, i.e. micro- and nano-scale geometry [11].

This paper first gets the DPL model equation through expanding DPL heat conduction expression to second order about time t coupled with energy balance equation. Then applying three orders finite difference method to the derived equation, the numerical results could be obtained and simulated with algorithm.

II. Mathematical Modeling

Considering the Dual-Phase-Lagging model expression as in (2-a) :

$$q(r,t+\tau_q) = -k\nabla T(r,t+\tau_T)$$

(2-a)

According to the Taylor series expansion, the heat flux q and the temperature gradient ∇T are expanded in second-order as in (2-b) :

$$q(r,t)+\tau_q\frac{\partial q(r,t)}{\partial t}+\frac{1}{2}\tau_q^2\frac{\partial q^2(r,t)}{\partial t^2} = -k\left[\nabla T(r,t)+\tau_T\frac{\partial(\nabla T(r,t))}{\partial t}+\frac{1}{2}\tau_T^2\frac{\partial^2(\nabla T(r,t))}{\partial t^2}\right]$$

(2-b)

Coupling with the local energy balance equation (without inner heat source) (2-c) :

$$\rho c\frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} \qquad \text{(2-c)}$$

Then the equation describing Dual-Phase-Lagging is obtained as in (2-d)

$$\frac{\partial T}{\partial t}+\tau_q\frac{\partial^2 T}{\partial t^2}+\frac{1}{2}\tau_q^2\frac{\partial^3 T}{\partial t^3} = \alpha\left[\frac{\partial^2 T}{\partial t\partial x}+\tau_T\frac{\partial^3 T}{\partial t\partial x^2}+\frac{1}{2}\tau_T^2\frac{\partial^4 T}{\partial t^2\partial x^2}\right]$$

(2-d)

Where: $\alpha = k/\rho c$

III. Numerical Analysis

The equation (2-d) is transformed into the node matrix equation (3-a,b), during which first-order space differentiation is dealt with backward-difference and the second-order space differentiation adopts central-difference.

$$\dot{T}_m+\tau_q\ddot{T}_m+\frac{1}{2}\tau_q^2\dddot{T}_m = \alpha\left[\frac{\dot{T}_m-\dot{T}_{m-1}}{\Delta x}+\tau_T\frac{\dot{T}_{m+1}-2\dot{T}_m+\dot{T}_{m-1}}{\Delta x^2}+\frac{1}{2}\tau_T^2\frac{\ddot{T}_{m+1}-2\ddot{T}_m+\ddot{T}_{m-1}}{\Delta x^2}\right]$$

(3-a)

$$\begin{bmatrix}\frac{\alpha}{\Delta x}-\frac{\tau_T}{\Delta x^2} & 1+\frac{2\tau_T}{\Delta x^2} & -\frac{\tau_T}{\Delta x^2}\end{bmatrix}\begin{bmatrix}\dot{T}_{m-1}\\ \dot{T}_m\\ \dot{T}_{m+1}\end{bmatrix} + \begin{bmatrix}-\frac{\tau_T^2}{2\Delta x^2} & \tau_q+\frac{\tau_T^2}{\Delta x^2} & -\frac{\tau_T^2}{2\Delta x^2}\end{bmatrix}\begin{bmatrix}\ddot{T}_{m-1}\\ \ddot{T}_m\\ \ddot{T}_{m+1}\end{bmatrix} + \begin{bmatrix}0 & \frac{\tau_q^2}{2} & 0\end{bmatrix}\begin{bmatrix}\dddot{T}_{m-1}\\ \dddot{T}_m\\ \dddot{T}_{m+1}\end{bmatrix} = 0$$

(3-b)

Afterwards applying the matrix assembly to the node matrix equation, the global stiffness matrix equation is expressed as follows (3-c):

$$[C]\left[\dot{T}_m^n\right]+[D]\left[\ddot{T}_m^n\right]+[K]\left[\dddot{T}_m^n\right] = 0 \qquad (1\le m\le M)$$

(3-c)

Matrices $[C]\,[D]\,[K]$ are of (M-2) rows and M columns.

$$\left[T_m^n\right] = \left[T_1^n \quad T_2^n \quad \cdots \quad T_M^n\right];$$

$$C(m,m) = \frac{\alpha}{\Delta x}-\frac{\tau_T}{\Delta x^2} \qquad (1\le m\le M-2);$$

$$C(m,m+1)=1+\frac{2\tau_T}{\Delta x^2} \quad (1\le m\le M-2);$$

$$C(m,m+2)=-\frac{\tau_T}{\Delta x^2} \quad (1\le m\le M-2);$$

$$D(m,m)=-\frac{\tau_T}{2\Delta x^2} \quad (1\le m\le M-2);$$

$$D(m,m+1)=\tau_q+\frac{\tau_T^2}{\Delta x^2} \quad (1\le m\le M-2);$$

$$D(m,m+2)=-\frac{\tau_T^2}{2\Delta x^2} \quad (1\le m\le M-2);$$

$$K(m,m+1)=\frac{\tau_q^2}{2} \quad (1\le m\le M-2);$$

The boundary conditions are:

$$m=1,\ \dot{T}_m^n=-\frac{1}{\rho c}\left(\frac{\partial q}{\partial x}\right)_{b1},\ \ddot{T}_m^n=0;$$

$$m=M,\ \dot{T}_m^n=-\frac{1}{\rho c}\left(\frac{\partial q}{\partial x}\right)_{b2},\ \ddot{T}_m^n=0;$$

Through substituting the boundary conditions into (3-c), the following global stiffness matrix equation with loaded vector [P] is gotten as (3-d):

$$[C]\left[\dot{T}_m^n\right]+[D]\left[\ddot{T}_m^n\right]+[K]\left[\dddot{T}_m^n\right]=[P] \quad (1\le m\le M)$$

(3-d)

$[C]\,[D]\,[K]$ are of (M-2) square matrices.

$[P]$ is of (M-2) column vector.

$$C(m,m)=1+\frac{2\tau_T}{\Delta x^2} \quad (1\le m\le M-2);$$

$$C(m,m+1)=-\frac{\tau_T}{\Delta x^2} \quad (1\le m\le M-3);$$

$$C(m,m-1)=\frac{\alpha}{\Delta x}-\frac{\tau_T}{\Delta x^2} \quad (2\le m\le M-2);$$

$$D(m,m)=\tau_q+\frac{\tau_T^2}{\Delta x^2} \quad (1\le m\le M-2);$$

$$D(m,m+1)=-\frac{\tau_T^2}{2\Delta x^2} \quad (1\le m\le M-3);$$

$$D(m,m-1)=-\frac{\tau_T^2}{2\Delta x^2} \quad (2\le m\le M-2);$$

$$K(m,m)=\frac{\tau_q^2}{2} \quad (1\le m\le M-2);$$

$$P(1,1)=-\frac{1}{\rho c}\left(\frac{\alpha}{\Delta x}-\frac{\tau_T}{\Delta x^2}\right)\left(\frac{\partial q}{\partial x}\right)_{b1},$$

$$P(M-2,1)=-\frac{1}{\rho c}\left(\frac{\alpha}{\Delta x}-\frac{\tau_T}{\Delta x^2}\right)\left(\frac{\partial q}{\partial x}\right)_{b2};$$

Applying backward-difference to the first-order and third-order time items and central difference to the second-order item, the node iteration equation is acquired as (3-e):

$$\frac{[K]}{\Delta t^3}T^{n+2}=\left(\frac{3[K]}{\Delta t^3}+\frac{[D]}{\Delta t^2}\right)T^{n+1}+\left(\frac{2[K]}{\Delta t^3}-\frac{2[D]}{\Delta t^2}+\frac{[C]}{\Delta t}\right)T^n+\left(\frac{[D]}{\Delta t^2}-\frac{[K]}{\Delta t^3}-\frac{[C]}{\Delta t}\right)T^{n-1}+[P]$$

(3-e)

$$T^n=\begin{bmatrix}T_2^n & T_3^n & \cdots & T_{M-2}^n & T_{M-1}^n\end{bmatrix}^T$$

The initial conditions are assumed as:

$$t=0,\ T^1=T^0,\ \dot{T}^1=\left(T^2-T^1\right)/\Delta t=0,$$

$$\ddot{T}^1=\left(T^3-2T^2+T^1\right)/\Delta t^2=0;$$

IV. RESULTS ILLUSTRATION

Through the above iteration which relates nodes temperature with coefficient matrix about τ_T, τ_q, α, ρ, c, Δx, Δt, $\partial q/\partial t\big|_{(b1,b2)}$, the temperature distribution at different time and position nodes are obtained, where the relaxation time τ_T, the heat flux relaxation time τ_q, heat diffusion ratio α, mass density ρ, specific heat c, space and time step Δx, Δt, as well as initial conditions T_1, T_2, T_3 and boundary conditions $\partial q/\partial t\big|_{b1}$, $\partial q/\partial t\big|_{b2}$ are determined according to the heat conditions medium properties and assuming heat shock environment. Furthermore, as for the resulting temperature, as for the resulting temperature distribution matrix T_m^n, the column T^n represents the different position nodes temperature variation at same time node, versus the row vector dictates the temperature changes with time at fixed position node. In addition, the vector [P] is the exerting heat shock load regarding heat flux variation ratio of two boundaries b_1, b_2, the elements of coefficient matrix [C] [D] [K] in relation with medium parameters. The space and time step should be appropriate in order to optimize the temperature distribution result, avoiding numerical disturbance.

V. CONCLUSION

Dual Phase Lagging heat conduction model is the indispensable description principal in some extreme heat input conditions, i.e. the surface disturbance with high frequencies [11]. The two introduced lagging physical parameters τ_T and τ_q associated with the intrinsic structure and properties of transferring medium, where τ_q reflects thermal inertia effect and confine heat conduction, whereas τ_T promote heat conduction process [12]. Through applying Taylor expansion to Dual Phase Lagging expression in second-order, coupled with local energy balance equation, more accurate mathematical equation which describes non-Fourier heat conduction is proposed. Space and time nodes are discrete, and temperature at each node could be acquired by numerical iteration. Dual Phase Lagging heat conduction equation could expresses the wave-like characteristics of heat and diffusion-like feature meanwhile. The higher the heat flux relaxation time, the more wave-like nature of equation, and the higher the temperature relaxation time, the stronger diffusion-like feature. The node discretion of space and time make assembled stiffness matrices obtained, and coefficient matrices are related with medium physical parameters. Be defined initial and boundary conditions, temperature distribution at each node could also be get iteration. The convergence and stability of results are another factor affecting temperature distribution. Based on the above analysis, space step and time step should be chosen in appropriate proportion and numerical value.

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