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LESLIE GOWER TYPE PREDATOR PREY MODEL WITH CONSTANT-EFFORT PREDATOR HARVESTING

Hazel Mae R. Diza[1, 2] & Joel M. Addawe[1]

[1]Department of Mathematics and Computer Science, College of Science, University of the Philippines Baguio, 2600 Baguio City, Philippines, hrdiza@up.edu.ph

[2]Mathematics Department, College of Computing Science, Pangasinan State University- Lingayen Campus, 240 Lingayen, Pangasinan, Philippines.

Abstract: In this study, we proposed to study the mathematical analysis of a Leslie Gower type predator prey model. The model considers the dynamics of a predator and prey populations with constant-effort harvesting applied in the predator population. We then computed and identified the existence of different equilibrium points of the model and investigated local stabilities of those points. We then proved that the system undergoes transcritical bifurcation.

Keywords: Predator-prey, constant-effort harvesting, equilibrium points

I. INTRODUCTION

In 1920, Lotka proposed a simple predator prey model of herbivores feeding on plants. Several years later, Volterra proposed the same model in fish population. Their model has become wide known for the name Lotka-Volterra model, or simply the predator prey model [1, 12]. Later a large number of literature and researches on the effect of predation, harvesting and competition on the ecological state have been made [14, 8, 3].

Modeling the dynamics of a predator prey has been greatly used in the management and exploitation of biological resources, specifically in the marine life. Marine life is a biological resource that provides important ecosystem services like food, medicine and livelihood [5, 18, 16, 6]. According to [2, 7], dynamics of predator prey model with harvesting plays a special role in the management of renewable resources. Due to human different background and community awareness, humans practice different ways of harvesting on animal predators that directly affects the prey population in a given available area. Appropriate harvesting method can result to an optimal resource management approach that may result to better co-existence of the predator and the prey population.

There are two kinds of harvesting regimes according to [13]: harvesting of constant-effort, and harvesting from constant-yield. Constant-effort harvesting is performed when proportion to a given population size is harvested, and harvested population using constant-yield harvesting is independent to the size of population.

In the study of May et al. [13], the authors describe the interaction between the predator and prey population where a different harvesting regime are considered and is illustrated by the given system:

$$\dot{x} = r_1 x\left(1-\frac{x}{K}\right) - axy - H_1, \qquad (1)$$
$$\dot{y} = r_2 y\left(1-\frac{y}{ab}\right) - H_2$$

where $x(t)$ and $y(t)$are variables that are assumed to be positive, in which x is the population density of the prey population and yis the population density of the predators for any non-negative value of t. H_1 - harvested density in the prey population and H_2- represent the harvested density in the predator population. r_1 - represents the intrinsic growth rate of the prey population while $r_2$represents the population intrinsic growth rate of predators. In the absence of predators, K then is considered the carrying capacity of the prey population. a is the parameter that represent the maximum amount the per capita reduction rate of the prey x(t) can obtain. b is the parameter value that represents the property of food for predators, whereas ab represents a prey-dependent carrying capacity for predators [13].

In a study of [11], the authors considered a case when independent harvesting are performed on predator-prey with prey-refuge, while in [20] did a study that considers a ratio dependent dynamics on the predator-prey when harvesting is introduced on the predator population. Several studies [19,21]considered different cases in using the Leslie Gower type predator prey model with different types of harvesting performed on either the predator or prey populations.

If we set $H_1 = H_2 = 0$, then system (1) is reduced to the model of a Leslie Gower predator prey. Analysis of Leslie Gower Model is shown in [8]. In 1980, Beddington and May considered the model of May et al. where constant effort of harvesting are considered, that is

$$\dot{x} = r_1 x\left(1-\frac{x}{K}\right) - axy - r_1h_1x, \quad \dot{y} = r_2 y\left(1-\frac{y}{ab}\right) - r_2h_2y \tag{2}$$

Where h_1 and h_2 are the harvesting efforts in the prey and predator populations respectively.

In the study of Bedding and Cooke [2], two cases were considered where the first case was on constant-yield harvesting applied in both the prey and predator populations, and the second was applying a type of on constant-yield harvesting on the prey and a different type of harvesting on the predators which was constant-effort. In the paper of Huang and Gong, the authors considered only the type of harvesting with constant-yield on predator population only. For [23] the authors only considered the constant-yield rate of harvesting performed on the prey population,

In this study, we look into the dynamics of a predator prey model in which we apply a constant-effort harvesting on the predator population only. We then consider the system (2) where $h_1 = 0$, that is,

$$\dot{x} = r_1 x\left(1-\frac{x}{K}\right) - axy, \quad \dot{y} = r_2 y\left(1-\frac{y}{ab}\right) - r_2h_2y. \tag{3}$$

In this study, we compute to show the existence of different equilibrium points, then analyze the stability of those points in a neighborhood. We then show several theorems to describe the dynamics of the proposed model. The paper ends by showing the occurrence of a transcritial bifurcation and by summarizing the effect of constant-effort harvesting in the predator population.

II. METHODOLOGY

We consider system (3) where $x(t)$ and $y(t)$ represent the prey and predator population at time t and h_2 is the harvesting effort in the predator population.

We simplify system (3) by the transformation in [10, 9], that is,

$$\bar{t} \to r_1 t, \quad \bar{x} \to \frac{x}{K}, \quad \bar{y} \to \frac{ay}{r_1}$$

$$\Rightarrow t = \frac{\bar{t}}{r_1}, \quad x = \bar{x}K, \quad y = \frac{r_1\bar{y}}{a}$$

$$\Rightarrow dt = \frac{d\bar{t}}{r_1}, \quad \frac{dx}{dt} = \dot{x} = \frac{Kr_1 d\bar{x}}{dt}, \quad \frac{dy}{dt} = \dot{y} = \frac{r_1^2 d\bar{y}}{adt}.$$

Substituting the above in the first equation of system (3), we get

$$\frac{kr_1 d\bar{x}}{d\bar{t}} = r_1(\bar{x}K)\left(1-\frac{\bar{x}K}{K}\right) - a\bar{x}K\left(\frac{r_1\bar{y}}{a}\right)$$

$$\Rightarrow \frac{d\bar{x}}{d\bar{t}} = \frac{r_1K\bar{x}(1-\bar{x}) - K\bar{x}r_1\bar{y}}{r_1K}$$

$$\Rightarrow \frac{d\bar{x}}{d\bar{t}} = \bar{x}(1-\bar{x}) - \bar{x}\bar{y}.$$

Similarly, the second equation of system (3) becomes

$$\frac{r_1^2 d\bar{y}}{ad\bar{t}} = r_2\left(\frac{r_1\bar{y}}{a}\right)\left(1-\frac{r_1\bar{y}}{abK\bar{x}}\right) - r_2h_2\left(\frac{r_1\bar{y}}{a}\right)$$

$$\Rightarrow \frac{d\bar{y}}{d\bar{t}} = \left(\frac{a}{r_1^2}\right)\left[\frac{r_1r_2\bar{y}}{a}\left(1-\frac{r_1\bar{y}}{abK\bar{x}}\right) - \frac{r_2h_2r_1\bar{y}}{a}\right]$$

$$\Rightarrow \frac{d\bar{y}}{d\bar{t}} = \bar{y}\left(\frac{r_2}{r_1} - \frac{r_2\bar{y}}{abK\bar{x}}\right) - \frac{r_2h_2\bar{y}}{r_1}.$$

For simplicity of notations, we then remove the bars, to have the system

$$\dot{x} = x(1-x) - xy, \quad \dot{y} = y\left(\frac{r_2}{r_1} - \frac{r_2 y}{abKx}\right) - \frac{r_2h_2y}{r_1}. \tag{4}$$

Let $\delta = \frac{r_2}{r_1}$, $\beta = \frac{r_2}{abK}$ be positive constants, then system (4) becomes

$$\dot{x} = x(1-x) - xy, \quad \dot{y} = y\left(\delta - \frac{\beta y}{x}\right) - h_2\delta y. \tag{5}$$

We now state some relevant theorems that will be used in this paper. The stability of system (5) depends directly on the values of its eigen values. In line with this, we state the results in [4] which is used to determine the stability of system (5) using their eigen values.

We consider eigen values λ_1 and λ_2 of the linear system corresponding to the locally linear system.

(a) If $\lambda_1 > \lambda_2 > 0$, then the locally linear system is an unstable node.

(b) If $\lambda_1 < \lambda_2 < 0$, then the locally linear system is asymptotically stable node.

(c) If $\lambda_1 < 0 < \lambda_2$, then the locally linear system is an unstable saddle point.

(d) If $\lambda_1, \lambda_2 = r \pm iu$ where $r > 0$, then the locally linear system is an unstable spiral point.

(e) If $\lambda_1, \lambda_2 = r \pm iu$ where $r < 0$, then the locally linear system is an asymptotically stable spiral point.

(f) If $\lambda_1 = iu$, $\lambda_2 = -iu$, then the locally linear system is center or spiral point.

Another theorem which will be used in our discussion is from [22] which is re--stated below:

Suppose, we assume that $E(0,0)$ is an isolated equilibrium point of $\dot{x} = P_2(x,y)$, $\dot{y} = y + Q_2(x,y)$ and

P_2, $Q_2$are analytic functions in $S_\delta(E)$ of orders no less than 2. Thus for δ sufficiently small, there exist an analytic function $\phi(x)$ satisfying $\phi(x) + Q_2(x, \phi(x)) = 0$ in $|x| < \delta$. Let $\psi(x) = P_2(x, \phi(x)) = a_m x^m + [x]_{m+1}$where $[x]_{m+1}$ represents the sum of those in $\psi(x)$ and $a_m \neq 0$, $m \geq 2$. Then the following properties are satisfied:

(a) If m is odd and $a_m > 0$, then E(0,0) is an unstable node.

(b) If m is odd and $a_m < 0$, then E(0,0) is a saddle point with its foru separatrices tending to E(0,0).

(c) If m is even, then E(0,0) is a saddle node. That is, $S_\delta(E)$ is divided into two different parts by separatrices that approach E(0,0) along the positive and negative y-axes. The first part is a parabolic sector and second part is composed of two hyperbolic sectors. Further, if $a_m > 0$(or < 0) then the parabolic sector is on the right (or left) halfplane.

The following theorem is from [17, 15] which we use to prove the existence of bifurcation in system (5).

Consider the system

$$\dot{x} = f((x,y),\mu),$$
$$\dot{y} = g((x,y),\mu) \tag{6}$$

Suppose that

$$f((x_0,y_0),\mu_0) = 0$$
$$g((x_0,y_0),\mu_0) = 0$$

and that the 2×2 matrix

$$A \equiv \begin{pmatrix} D_x[f(x,y,\mu)] & D_y[f(x,y,\mu)] \\ D_x[g(x,y,\mu)] & D_y[g(x,y,\mu)] \end{pmatrix}_{((x_0,y_0),\mu_0)}$$

can be shown to have a simple eigenvalue$\lambda = 0$ with the corresponding computed eigenvector given by$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$and that$A^T$has a corresponding eigenvector$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$for a given eigenvalue$\lambda = 0$. Assume also that matrixA hasa number kof eigen values with the real parts being negative, and$(n-k-1)$eigenvalues whose real parts being positive, such that:

$$\mathbf{w}^T \begin{pmatrix} D_\mu[f((x,y),\mu)] \\ D_\mu[g((x,y),\mu)] \end{pmatrix}_{((x_0,y_0),\mu_0)} = 0$$

$$\mathbf{w}^T \begin{pmatrix} D_x\left[D_\mu[f((x,y),\mu)]\right] v_1 + D_y\left[D_\mu[f((x,y),\mu)]\right] v_2 \\ D_x\left[D_\mu[g((x,y),\mu)]\right] v_1 + D_y\left[D_\mu[g((x,y),\mu)]\right] v_2 \end{pmatrix}_{((x_0,y_0),\mu_0)} \neq 0$$

$$\mathbf{w}^T \begin{pmatrix} \frac{\partial^2 f((x,y),\mu)}{\partial x^2} v_1^2 + 2\frac{\partial^2 f((x,y),\mu)}{\partial x \partial y} v_1 v_2 + \frac{\partial^2 f((x,y),\mu)}{\partial y^2} v_2^2 \\ \frac{\partial^2 f((x,y),\mu)}{\partial x^2} v_1^2 + 2\frac{\partial^2 f((x,y),\mu)}{\partial x \partial y} v_1 v_2 + \frac{\partial^2 f((x,y),\mu)}{\partial y^2} v_2^2 \end{pmatrix}_{((x_0,y_0),\mu_0)} \neq 0$$

Then system (6) can then be shown to illustrate a transcritical bifurcation at the equilibrium point (x_0, y_0) as the parameter μ varies through the bifurcation value $\mu = \mu_0$.

III. RESULTS

A. *Existence of Equilibria*

From the biological viewpoint, we are only interested in the values of predator population density, that is, $y(t) \geq 0$. Similarly, we consider the positive values of the prey population density, that is, $x(t) > 0$. We denote this domain as $\Omega = x(t) > 0, y(t) \geq 0\}$.

To get the equilibrium points, we compute the system

$$\dot{x} = x(1-x) - xy = 0,$$
$$\dot{y} = y\left(\delta - \frac{\beta y}{x}\right) - h_2\delta y = 0. \tag{7}$$

For values of x and y in Ω. Computing for x and y, we obtain the following theorem:

Theorem 1

For all parameters assumed to be positive, the equilibrium points of system (5) are as follows:

(a) There is a unique boundary equilibrium point $E_0 = (1,0)$ in Ω.

(b) If $h_2 > 1$, there is no positive equilibrium point in Ω.

(c) If $h_2 < 1$, there is a unique equilibrium point $E_1 = \left(\frac{\beta}{\delta(1-h_2)+\beta}, \frac{\delta(1-h_2)}{\delta(1-h_2)+\beta}\right)$ in Ω.

Proof. It is easy to see that $E_0(1,0)$ is an equilibrium point of the system. For possible positive equilibrium$1 - x - y = 0$ and $\delta - h_2\delta - \frac{\beta y}{x} = 0$. This implies that $y = 1 - x$. We substitute this in $\delta - h_2\delta - \frac{\beta y}{x} = 0$ and have $\delta - h_2\delta - \frac{\beta(1-x)}{x} = 0$. Solving for x and y, we have $(x_1, y_1) = \left(\frac{\beta}{\delta(1-h_2)+\beta}, \frac{\delta(1-h_2)}{\delta(1-h_2)+\beta}\right)$. If $h_2 > 1$, then there are three scenarios in the values of x_1 and y_1. First if $\delta(1-h_2) > \beta$, then $x_1 < 0$. If $\delta(1-h_2) < \beta$, then $y_1 < 0$. Also, if $\delta(1-h_2) = \beta$, then $\beta < 0$ which is a contradiction to our assumption that β is a positive constant. Hence, this implies that if $h_2 > 1$ we cannot have (x_1, y_1) in Ω. Suppose now that $h_2 < 1$, then there is a unique positive equilibrium point $E_1 = (x_1, y_1)$ in Ω. This completes the proof.

Stability of the Equilibrium Points

Stability of E_0

Theorem 2

For all positive parameters, system (5) has unique boundary equilibrium $E_0 = (1,0)$ in Ω and

(a) If $h_2 < 1$, then E_0 is a hyperbolic unstable saddle point.
(b) If $h_2 < 1$, then E_0 is a hyperbolic stable node.
(c) If $h_2 = 1$, then E_0 is a saddle node, that is, a neighborhood of E_0 is divided into two parts by two saparatrices that tend to E_0 along the upside and underneath of E_0. The lower half plane consists of the parabolic sector and the upper half plane consist two hyperbolic sectors.

Proof**.** The Jacobian matrix of system (5) is given by

$$J(x,y) = \begin{pmatrix} 1-2x-y & -x \\ \frac{\beta y^2}{x^2} & \delta - h_2\delta - \frac{2\beta y}{x} \end{pmatrix}.$$

At E_0, the Jacobian matrix becomes$J(E_0) = \begin{pmatrix} -1 & -1 \\ 0 & \delta - h_2\delta \end{pmatrix}$. By simple calculations, then eigenvalues are $\lambda = -1$ and $\lambda = \delta - h_2\delta$. By [4] if $\delta - h_2\delta > 0$, that is, $h_2 < 1$, then E_0 is a hyperbolic saddle. If $\delta - h_2\delta < 0$, that is, $h_2 > 1$, then E_0 is a hyperbolic stable node.

If $h_2 = 1$, then one eigenvalue is negative and the other is zero. The preceding argument is the same from [9] and [10]. First, we transform E_0 to the origin by the translation $(X,Y) = (x-1, y)$, that is, $X = x - 1 \Rightarrow x = X + 1$ and $Y = y$. System (5) becomes

$$\dot{X} = (X+1)\big(1-(X+1)\big) - (X+1)Y,$$
$$\dot{Y} = Y\left(\delta - \frac{\beta Y}{X+1}\right) - h_2\delta Y.$$

Simplifying, we have,

$$\dot{X} = -X - Y - X^2 - XY,$$
$$\dot{Y} = \delta Y - h_2\delta Y - \frac{\beta Y^2}{X+1}. \quad (8)$$

Note that $\frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots$ is the power series expansion of $\frac{1}{x+1}$. Expanding system (8) above in power series up to the 4th order, we obtain

$$\dot{X} = -X - Y - X^2 - XY,$$
$$\dot{Y} = \delta Y - h_2\delta Y - \beta Y^2(1 - X + X^2 - X^3 + X^4) + Q_1(X,Y)$$

which implies that

$$\dot{X} = -X - Y - X^2 - XY,$$
$$\dot{Y} = Y(\delta - h_2\delta) + Y^2(-\beta) + XY^2(\beta) + X^2Y^2(-\beta) + Q_1(X,Y).$$

Recall that $\delta = h_2\delta$, hence we have

$$\dot{X} = -X - Y - X^2 - XY,$$
$$\dot{Y} = -\beta Y^2 + \beta XY^2 - \beta X^2Y^2 + Q_1(X,Y).$$
(9)

where $Q_1(X,Y)$ is a series of order greater than 4.

We now find the matrix T that makes system (9) into normal form. If $\lambda = -1$, we get $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which implies that $Y = 0$. This means that if $\lambda = -1$, one eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $\lambda = 0$, we get $\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which implies that $X + Y = 0$. Hence an eigenvector for the system if $\lambda = 0$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Let $T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ be the matrix that transform system (9) into normal form. Accordingly, we have

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow X = x - y,\ Y = y$ and $\dot{X} = \dot{x},\ \dot{Y} = \dot{y}$.

System (9) becomes

$$\dot{x} = -x - x^2 + xy - \beta y^2 + \beta y^2 x - \beta y^3 - \beta y^2 x^2 + 2\beta y^3 x - \beta y^4 + P_2(x,y),$$
$$\dot{y} = -\beta y^2 + \beta y^2 x - \beta y^3 - \beta y^2 x^2 + 2\beta y^3 x - \beta y^4 + Q_2(x,y)$$

where $P_2(x,y)$ and $Q_2(x,y)$ are series of order greater than 4. We then introduce a a new time variable $\tau = -t$, we have

$$\dot{x} = x + x^2 - xy + \beta y^2 - \beta y^2 x + \beta y^3 + \beta y^2 x^2 - 2\beta y^3 x + \beta y^4 + P_3($$
$$\dot{y} = \beta y^2 - \beta y^2 x + \beta y^3 + \beta y^2 x^2 - 2\beta y^3 x + \beta y^4 + Q_3(x,y)$$
(10)

where $P_3(x,y)$ and $Q_3(x,y)$ are series of order greater than four.

Applying the Theorem from [22] to system (10) and noting that $m = 2$ which is even, we have E_0 a saddle node. Also, $\beta > 0$ which implies that the parabolic sector is in the upper half plane where the orbit of time going into the opposite direction.

Stability of E_1

We then show the stability of E_1.

Theorem 3

For all parameters assumed to be positive and $h_2 < 1$, system (5) has a unique positive equilibrium $E_1 = \left(\frac{\beta}{\delta(1-h_2)+\beta}, \frac{\delta(1-h_2)}{\delta(1-h_2)+\beta}\right)$ in Ω. Moreover, E_1 is locally asymptotically stable and

(a) If $TR^2 - 4\delta(1-h_2) \geq 0$, then E_1 is a hyperbolic node
(b) If$TR^2 - 4\delta(1-h_2) < 0$, then E_1 is a hyperbolic spiral point

where TR is the trace of the Jacobian matrix of system (5).

Proof. Recall that the Jacobian matrix of system (5) is given by

$$J(x,y) = \begin{pmatrix} 1-2x-y & -x \\ \frac{\beta y^2}{x^2} & \delta - h_2\delta - \frac{2\beta y}{x} \end{pmatrix}.$$

At E_1, we have

$$J(E_1) = \begin{pmatrix} -x & -x \\ \frac{\beta y^2}{x^2} & -\frac{\beta y}{x} \end{pmatrix} = \begin{pmatrix} -x & -x \\ \frac{(\delta - h_2\delta)^2}{\beta} & -(\delta - h_2\delta) \end{pmatrix}.$$

We compute the determinant of this Jacobian matrix, and have

$Det\big(J(E_1)\big) = x(\delta - h_2\delta) + \frac{x(\delta - h_2\delta)^2}{\beta} = \left(\frac{\beta}{\delta - h_2\delta + \beta}\right)(\delta -$ h2δ+βδ−h2+βδ−h2δ2β

$\Rightarrow Det\big(J(E_1)\big) = \left(\frac{\delta - h_2\delta}{\delta - h_2\delta + \beta}\right)(\beta + \delta - h_2\delta) = \delta - h_2\delta = \delta(1-h_2) > 0.$

Similarly, we compute for the trace (TR) of the Jacobian matrix and we have

$TR\big(J(E_1)\big) = -x - \frac{\beta y}{x} = \frac{-\beta}{\delta - h_2\delta + \beta} - \beta\left(\frac{\delta - h_2\delta}{\beta}\right) = \frac{-\beta}{\delta - h_2\delta + \beta} - (\delta - h_2\delta) < 0.$

Note that the eigenvalues of $J(E_1)$ are

$\lambda_{1,2} = \frac{TR(J(E_1)) \pm \sqrt{(TR(J(E_1)))^2 - 4\delta(1-h_2)}}{2}$.

From a theorem in [4] it is easy to see that since $TR(J(E_1)) < 0$, then E_1 is either a stable node or stable spiral node. This can be determined by computing the value of $TR^2 - 4\delta(1-h_2)$. Hence this ends the proof of the theorem.

Bifurcation Analysis

The next theorem states the existence of transcritical bifurcation in system (5).

Theorem 4

System (5) undergoes transcritical bifurcation at $E_0(1,0)$ as h_2 changes through $h_2 = 1$.

Proof. Note that as the parameter h_2 passes through $h_2 = 1$, the stability of the equilibrium points E_0 changes from unstable to stable in system (5). Hence this implies that $h_2 = 1$ is a bifurcation values of the system. Using Sotomayor's theorem into system (5), we suppose that

A

$$= \begin{pmatrix} D_x[x(1-x)-xy] & D_y[x(1-x)-xy] \\ D_x\left[y\left(\delta - \frac{\beta y}{x}\right) - h_2\delta y\right] & D_y\left[y\left(\delta - \frac{\beta y}{x}\right) - h_2\delta y\right] \end{pmatrix}_{(E_0,1)}$$

$$\Rightarrow A = \begin{pmatrix} 1-2x-y & -x \\ \frac{\beta y^2}{x^2} & \delta - h_2\delta - \frac{2\beta y}{x} \end{pmatrix}_{(E_0,1)}$$

$$= \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

We can easily see from the above previous statement that $\lambda = 0$ is an eigenvalue of A and its corresponding computed eigenvector is $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Similarly, $A^T = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$ can easily be shown to have an eigenvalue $\lambda = 0$ with the computed corresponding eigenvector $\boldsymbol{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Computing we have

$$\boldsymbol{w}^T \begin{pmatrix} D_{h_2}[x(1-x)-xy] \\ D_{h_2}\left[y\left(\delta - \frac{\beta y}{x}\right) - h_2\delta y\right] \end{pmatrix}_{(E_0,1)} =$$

$$(0\ \ 1)\begin{pmatrix} 0 \\ -\delta y \end{pmatrix}_{(E_0,1)} = (0\ \ 1)\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0,$$

$$\boldsymbol{w}^T \begin{pmatrix} D_x\left[D_{h_2}[x(1-x)-xy]\right]v_1 + D_y\left[D_{h_2}[x(1-x)-xy]\right]v_2 \\ D_x\left[D_{h_2}\left[y\left(\delta - \frac{\beta y}{x}\right) - h_2\delta y\right]\right]v_1 + D_y\left[D_{h_2}\left[y\left(\delta - \frac{\beta y}{x}\right) - h_2\delta y\right]\right]v_2 \end{pmatrix}_{(E_0,1)}$$

$$= (0\ \ 1)\begin{pmatrix} 0v_1 + 0v_2 \\ 0v_1 + (-\delta v_2) \end{pmatrix} = (0\ \ 1)\begin{pmatrix} 0 \\ -\delta \end{pmatrix} = -\delta \neq 0$$

$$w^T \begin{pmatrix} \frac{\partial^2[x(1-x)-xy]}{\partial x^2}v_1^2 + 2\frac{\partial^2[x(1-x)-xy]}{\partial x\partial y}v_1v_2 + \frac{\partial^2[x(1-x)-xy]}{\partial y^2}v_2^2 \\ \frac{\partial^2\left[y\left(\delta - \frac{\beta y}{x}\right) - h_2\delta y\right]}{\partial x^2}v_1^2 + 2\frac{\partial^2\left[y\left(\delta - \frac{\beta y}{x}\right) - h_2\delta y\right]}{\partial x\partial y}v_1v_2 + \frac{\partial^2\left[y\left(\delta - \frac{\beta y}{x}\right) - h_2\delta y\right]}{\partial y}v_2^2 \end{pmatrix}_{(E_0,1)}$$

$$= (0\ \ 1)\begin{pmatrix} -2v_1^2 + 2(-1)v_1v_2 + 0v_2^2 \\ -\frac{2\beta y^2 x}{x^4}v_1^2 + 2\frac{2\beta y}{x^2}v_1v_2 - \frac{2\beta}{x}v_2^2 \end{pmatrix}_{(E_0,1)}$$

$$= (0\ \ 1)\begin{pmatrix} 0 \\ -2\beta \end{pmatrix} = -2\beta \neq 0$$

Hence, by Sotomayor's Theorem [17, 15], system (5) undergoes a transcritical bifurcation.

IV. CONCLUSIONS

The considered system (5) is shown to have a unique boundary equilibrium $E_0(1,0)$. Theorem 1 states that if $h_2 > 1$, then we see that there is no positive equilibrium point, hence the only equilibrium point is the boundary equilibrium E_0. Using the parameters of the original system (3) we get $E_0 = (K, 0)$. As shown in Theorem 2, if $h_2 > 1$ then E_0 is a hyperbolic stable node which means that if the harvesting effort in the predator population is greater than 1 then the predator population will approach extinction and the prey population and tends to approach the carrying capacity K.

Similarly, if $h_2 = 1$, then the only equilibrium point is E_0. As shown in Theorem 2, if $h_2 = 1$, then E_0 is a saddle node. This implies that if the harvesting effort h_2 in the predator population is equal to one then the predator population will go extinct and the prey population approaches K.

In Theorem 1, if $h_2 < 1$, we have shown the existence of a positive equilibrium $E_1 = \left(\frac{\beta}{\delta(1-h_2)+\beta}, \frac{\delta(1-h_2)}{\delta(1-h_2)+\beta}\right)$. Using the same parameters used in system (3), we then have $E_1 = \left(\frac{r_1 K}{r_1 + abK - abK h_2}, \frac{r_1 bK - r_1 bK h_2}{r_1 + abK - abK h_2}\right)$. From Theorem 3, E_1 is a stable equilibrium point. Hence if the harvesting effort h_2 is less than one then there exist coexistence between the predator and prey population.

Furthermore, we have shown the existence of transcritical bifurcation using Sotomayor's theorem. The existence of this bifurcation will lead to changes in the dynamics of the system, specifically, this shows that too much exploitation of available resources can lead to extinction.

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REFERENCES

[1] Baca¨er, N. 2011. "A short history of mathematical population dynamics", Springer Science & Business Media.

[2] Beddington, J. Cooke, J. 1982. "Harvesting from a prey-predator complex", Ecological Modelling143-4155–177.

[3] Beddington, J. May, R. 1980. "Maximum sustainable yields in systems subject to harvesting at more than one trophic level", Mathematical Biosciences513-4261–281.

[4] Boyce, WE., DiPrima, RC. Haines, CW. 1969. "Elementary differential equations and boundary value problems",(9). Wiley New York.

[5] Clark, C. 1990. "Mathematical bioeconomics".

[6] Edelstein-Keshet, L. 1988. "Mathematical models in biology(46)", Siam.

[7] Gong, Yj. Huang, Jc. 2014. "Bogdanov-Takens bifurcation in a Leslie-Gower predator-prey model with prey harvesting", Acta Mathematicae Applicatae Sinica, EnglishSeries301239–244.

[8] Hsu, SB. Huang, TW. 1995. "Global stability for a class of predator-prey systems", SIAM Journal on Applied Mathematics553763–783.

[9] Hu, D. Cao, H. 2017. "Stability and bifurcation analysis in a predator–prey system with Michaelis–Menten type predator harvesting", Nonlinear Analysis: Real World Applications3358–82.

[10] Huang, J., Gong, Y. Ruan, S. 2013. "Bifurcation analysis in a predatorprey model with constant-yield predator harvesting", Discrete Contin. Dynam. Syst. Ser. B182101–2121.

[11] Kar, TK. 2006. "Modelling and analysis of a harvested preypredator system incorporating a prey refuge", Journal of Computational and Applied Mathematics 185 (2006) 19 3318519–33.

[12] Kofoid, C. 1925. "Elements of physical biology",American Public Health Association.

[13] Mahapatra, D. K. (2014). Neuraminidase Inhibitors – For Effective Treatment of Influenza 22 | IJPRT | January – March Mahapatra et al / International Journal of Pharmacy Research & Technology 2014, 4(1), 22–31.

[14] Mukhopadhyay, B. Bhattacharyya, R. 2016. "Effects of harvesting and predator interference in a model of two predators competing for a single prey", Applied Mathematical Modelling4043264–3274.

[15] Perko, L. 2013. "Differential equations and dynamical systems(7)", Springer Science & Business Media.

[16] Singh, MK., Bhadauria, B. Singh, BK. 2016. "Qualitative Analysis of a Leslie-Gower Predator-Prey System with Nonlinear Harvesting in Predator", International Journal of Engineering Mathematics2016.

[17] Sotomayor, J. 1973. "Generic bifurcations of dynamical systems", Dynamical systems (561–582). Elsevier.

[18] Sumich, JL. Morrissey, JF. 2004. "Introduction to the biology of marine life", Jones & Bartlett Learning.

[19] Toaha, S. Azis, MI. 2018. "Stability and Optimal Harvesting of Modified Leslie-Gower Predator Prey Model",J. Phys.: Conf. Ser.979012069

[20] Xiao, D., Li, W. Han, M. 2006. "Dynamics in a ratio-dependent predatorprey model with predator harvesting", J. Math. Anal. Appl.304440314–29.

[21] Yang, R., Zhang, C. 2017. "Dynamics in a diffusive modified Leslie-Gower predatory-prey model with time delayand prey harvesting", Nonlinear Dynamics. 87: 863. https://doi.org/10.1007/s11071-016-3084-7

[22] Zhi-Fen, Z., Tong-Ren, D., Wen-Zao, H. Zhen-Xi, D. 2006. "Qualitative theory of differential equations(101)", American Mathematical Soc.

[23] Zhu, C. Lan, K. 2010. "Phase portraits, Hopf bifurcations and limit cycles of Leslie-Gower predator-prey systems with harvesting rates", Discrete & Continuous Dynamical Systems-B141289–306.