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EXPLICIT GROUP ITERATIVE METHODS IN THE SOLUTION OF TWO DIMENSIONAL TIME-FRACTIONAL DIFFUSION-WAVES EQUATION

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Abstract: In this paper, we present the preliminary study of the formulation of fractional explicit group (FEG) and fractional explicit de-coupled group (FEDG) iterative methods in solving the two dimensional second order diffusion wave equation of fractional order. Both FEG and FEDG iterative methods are derived from the fractional standard and fractional rotated five points Crank-Nicolson discretizations respectively. Their computational complexity is presented and numerical experiments are conducted to demonstrate the efficiency and adeptness of the newly developed explicit group formulations in terms of CPU timings and total number of operations. AMS Subject Classification: 65N14

Keywords and Phrases: FSP, FRP, FEG, FEDG, time-fractional diffusion-wave equation, Caputo's fractional derivative.

I. INTRODUCTION

Fractional differential equation (FDE) which is the generalization of the integer ordered differential equation have gained considerable significance in the field of engineering [6], [20], quantum mechanics [15], hydrology [10], visco-elasticity [5], [21] bio science [19], control system [1] and other sciences [29]. Due to non-local property and more realistic natural phenomena, several mathematical problems can be solved in these _led of studies by utilizing the FDE in terms of time-fractional, space-fractional and time-space- fractional differential equations. But to discover the exact analytic solutions of FDE is complex task. Therefore, it is essential to develop the stable, precise and well-organized numerical approximations to the exact analytic solutions [22]. Several scholars efficiently solved the time-fractional, space-fractional and time-space-fractional for FDE using many proficient methods such as Galerkin spectral method [24], [25], homotopy perturbation method [27], compact alternating direct implicit method [28], singular boundary method [4], local radial point interpolation method [26] and fourth order finite difference method [16]. In both cases whether equations is discretized fractionally or non-fractionally (positive integer ordered), the numerical solution of the discretized equation along with the usage of finite difference approximation generates system of linear equations. This system of linear equations impelled the development of significant number of iterative methods to achieve the convergence. In case both cases, many iterative methods have been suggested by the different scholars such as Evans [14] proposed the group explicit iterative method in the solution of large linear systems. Othman and Abdullah [23] utilized the modified explicit group iterative method to solve the Poisson differential equation while Evans and Sahimi [11] recommended the alternative group explicit (AGE) iterative method to solve one and two dimensional parabolic problems and they also utilized the same method for hyperbolic partial differential equations. Evans and Changjun [12] used AGE iterative method in linear algebra, Evans and Yousif [13] utilized the explicit group iterative methods for spare system of linear equation in the solution of elliptic partial differential equation. Ali and Ng [3] developed modified explicit de-coupled group iterative method in the solution of 2-D elliptic equation. Meanwhile, Ali and Kew [2] considered the concept of modified explicit de-coupled group iterative method in the solution of 2-D telegraph PDEs. The new explicit group iterative method in the solution of three dimensional hyperbolic telegraph equations was also formulated by Kew and Ali [17]. Recently, Balasim and Ali [7], [8], [9] utilized fractional standard point (FSP), fractional rotated point (FRP), fractional explicit group (FEG), fractional modified explicit group (FMEG), fractional explicit de-coupled group (FEDG) and fractional modified explicit de-coupled group(FMEDG) iterative methods for the solution of diffusion differential equation of fractional order and find significant results in terms of execution of time, number of iterations and total number of operations in solving the time-fractional diffusion equation.

II. METHODOLOGY

In this paper, we proposed the formulation of explicit group methods in solving the two dimensional second order time-fractional diffusion-wave equations,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a(\mathrm{x,y,t})\frac{\partial^2 u}{\partial x^2} + \mathrm{b(x,y,t)}\frac{\partial^2 u}{\partial y^2} + \mathrm{f(x,y,t)} \quad \ldots\ldots (1)$$

where $1 < \alpha < 2, a(x,y,t)$ and $b(x,y,t)$ are the variable co-efficients and x, y and t represent the spatial and temporal characterizations respectively and $f(x,y,t)$ is the source term with initial and boundary conditions:

$u(x,y,0) = \phi(x,y),\ ut(x,y,0) = 0$

$u(0,y,t) = g_1(y,t), \quad u(L,y,t) = g_2(y,t)$

$u(x,0,t) = g_3(x,t), \quad u(x,L,t) = g_4(x,t)$

Where

$\omega = \{(x,y,t) / \quad 0 < x,y < L, \quad 0 \le t \le T\}$

The discrete derivation of Caputo's time-fractional derivative of order α is defined as,

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{\partial u^m(x,y,t)}{\partial u\xi^m}\frac{d\xi}{(t-\xi)^{\alpha+1-m}} \quad \ldots\ldots (2)$$

Where $m-1 < \alpha < m$.

Discretize the solution domain by defining $t_k = k\tau, k = 0,1,2 \ldots N, x_i = i\Delta x, i = 0,1,2 \ldots M_x, y_j = j\Delta y, j = 0,1,2 \ldots M_y$

where $\tau = \frac{T}{N}$, $\Delta x = \frac{L}{M_x}$ and $\Delta y = \frac{L}{M_y}$.

Let $U_{i,j}^k$ be the exact solution and $u_{i,j}^k$ be the approximate solution of FDE (1) at the grid point $(x_i, y_j, t_{k+1},)$.Consider, $f(x_i, y_j, t_k,) = f_{i,j}^k$, $a(x_i, y_j, t_k) = a_{i,j}^k$, $b(x_i, y_j, t_k) = b_{i,j}^k$

Utilizing the second order time differential operator in equation (2), we get the following α order time-fractional approximation [18] at the point $(x_i, y_j, t_{k+1},)$,

$$\frac{\partial^\alpha u(x_i,y_j,t_{k+1},)}{\partial t^\alpha} = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{s=0}^{k} b_s\left[u_{i,j}^{k-s+1} - 2u_{i,j}^{k-s} + u_{i,j}^{k-s-1}\right] \quad \ldots\ldots (3)$$

Where $b_s = (s+1)^{(2-\alpha)} - (s)^{(2-\alpha)},\ s = 0,1,2,\ldots.n$

Fractional Standard Point (FSP) Iterative Scheme

By using Caputo's fractional derivative of order α in equation (3) and standard Crank-Nicolson finite difference approximation in equation (1), the following fractional standard point (FSP) Crank-Nicolson scheme is obtained,

$$u_{i,j}^{k+1} = \frac{1}{(1+r_1+r_2)}\Bigg[\frac{r_1}{2}\left(u_{i-1,j}^{k+1} + u_{i+1,j}^{k+1}\right) + \frac{r_2}{2}\left(u_{i,j-1}^{k+1} + u_{i,j+1}^{k+1}\right) + \frac{r_1}{2}\left(u_{i-1,j}^{k} + u_{i+1,j}^{k}\right) + \frac{r_2}{2}\left(u_{i,j-1}^{k} + u_{i,j+1}^{k}\right) + (2 - r_1 - r_2 - b_1)u_{i,j}^k + (2b_k - b_{k-1})u_{i,j}^0 - b_k u_{i,j}^1 - \sum_{s=1}^{k-1}(b_{s-1} + 2b_s + b_{s+1})u_{i,j}^{k-s} + \tau^\alpha\Gamma(3-\alpha)f_{i,j}^k\Bigg] \quad \ldots\ldots (4)$$

For all $i = 1,2 \ldots M_x y_j, j = 1,2 \ldots M_y$ and $k = 0,1,2 \ldots N$

Where $\mu_1 = \frac{\tau^\alpha}{(\Delta x)^2}$, $\mu_1 = \frac{\tau^\alpha}{(\Delta y)^2}$, $r_1 = \mu_1\tau^\alpha\Gamma(3-\alpha)a_{i,j}^k$, $r_2 = \mu_2\tau^\alpha\Gamma(3-\alpha)b_{i,j}^k$

Fractional Rotated Point (FRP) Iterative Scheme

Utilizing the Caputo's fractional derivative of order α in (3) and rotated Crank-Nicolson finite difference approximation at an angle 45^o to the standard mesh in equation (1), the following fractional rotated point (FRP) Crank-Nicolson scheme is obtained,

$$u_{i,j}^{k+1} = \frac{1}{(1+r_1/2+r_2/2)}\Bigg[\frac{r_1}{4}\left(u_{i-1,j+1}^{k+1} + u_{i+1,j-1}^{k+1}\right) + \frac{r_2}{4}\left(u_{i-1,j-1}^{k+1} + u_{i+1,j+1}^{k+1}\right) + \frac{r_1}{4}\left(u_{i-1,j+1}^{k} + u_{i+1,j-1}^{k}\right) + \frac{r_2}{4}\left(u_{i-1,j-1}^{k} + u_{i+1,j+1}^{k}\right) + \left(2 - \frac{r_1}{2} - \frac{r_2}{2} - b_1\right)u_{i,j}^k + (2b_k - b_{k-1})u_{i,j}^0 - b_k u_{i,j}^1 - \sum_{s=1}^{k-1}(b_{s-1} + 2b_s + b_{s+1})u_{i,j}^{k-s} + \tau^\alpha\Gamma(3-\alpha)f_{i,j}^k\Bigg] \quad \ldots\ldots (5)$$

For all $i = 1,2 \ldots M_x y_j, j = 1,2 \ldots M_y$ and $k = 0,1,2 \ldots N$

Fractional Explicit Group (FEG) Iterative Scheme

In developing the FEG iterative scheme, we divide the whole solution domain into groups of four points. In doing so the group of points will be treated as single point like the standard or rotated point finite difference approximations by reducing the arithmetic operations and lapse time per iteration.

Apply equation (4) on a group of four points will result 4 x 4 system of equations,

$$\begin{pmatrix} k_1 & k_2 & k_3 & k_4 \\ k_2 & k_1 & k_4 & k_3 \\ k_3 & k_4 & k_1 & k_2 \\ k_4 & k_3 & k_2 & k_1 \end{pmatrix} \begin{pmatrix} u_{i,j}^{k+1} \\ u_{i+1,j}^{k+1} \\ u_{i+1,j+1}^{k+1} \\ u_{i,j+1}^{k+1} \end{pmatrix} = \begin{pmatrix} rhs_{i,j} \\ rhs_{i+1,j} \\ rhs_{i+1,j+1} \\ rhs_{i,j+1} \end{pmatrix}$$

Where$k_1 = 1 + r_1 + r_2$, $k_2 = -r_1/2$, $k_3 = 0$ and $k_4 = -r_2/2$

$$\begin{aligned} rhs_{i,j} = {} & \frac{r_1}{2} u_{i-1,j}^{k+1} + \frac{r_2}{2} u_{i,j-1}^{k+1} + \frac{r_1}{2}\left(u_{i-1,j}^{k} + u_{i+1,j}^{k}\right) \\ & + \frac{r_2}{2}\left(u_{i,j-1}^{k} + u_{i,j+1}^{k}\right) \\ & + (2 - r_1 - r_2 - b_1) u_{i,j}^{k} \\ & + (2b_k - b_{k-1}) u_{i,j}^{0} \\ & - b_k u_{i,j}^{1} - \sum_{s=1}^{k-1} (b_{s-1} + 2b_s \\ & + b_{s+1}) u_{i,j}^{k-s} + \tau^{\alpha} \Gamma(3-\alpha) f_{i,j}^{k}] \end{aligned}$$

$$\begin{aligned} rhs_{i+1,j} = {} & \frac{r_1}{2} u_{i,j}^{k+1} + \frac{r_2}{2} u_{i+1,j-1}^{k+1} + \frac{r_1}{2}\left(u_{i,j}^{k} + u_{i+2,j}^{k}\right) \\ & + \frac{r_2}{2}\left(u_{i+1,j-1}^{k} + u_{i+1,j+1}^{k}\right) \\ & + (2 - r_1 - r_2 - b_1) u_{i+1,j}^{k} \\ & + (2b_k - b_{k-1}) u_{i+1,j}^{0} \\ & - b_k u_{i+1,j}^{1} - \sum_{s=1}^{k-1} (b_{s-1} + 2b_s \\ & + b_{s+1}) u_{i+1,j}^{k-s} + \tau^{\alpha} \Gamma(3-\alpha) f_{i+1,j}^{k}] \end{aligned}$$

$$\begin{aligned} rhs_{i+1,j+1} = {} & \frac{r_1}{2} u_{i,j+1}^{k+1} + \frac{r_2}{2} u_{i+1,j}^{k+1} + \frac{r_1}{2}\left(u_{i,j+1}^{k} + u_{i+2,j+1}^{k}\right) \\ & + \frac{r_2}{2}\left(u_{i+1,j}^{k} + u_{i+1,j+2}^{k}\right) \\ & + (2 - r_1 - r_2 - b_1) u_{i+1,j+1}^{k} \\ & + (2b_k - b_{k-1}) u_{i+1,j+1}^{0} \\ & - b_k u_{i+1,j+1}^{1} - \sum_{s=1}^{k-1} (b_{s-1} + 2b_s \\ & + b_{s+1}) u_{i+1,j+1}^{k-s} \\ & + \tau^{\alpha} \Gamma(3-\alpha) f_{i+1,j+1}^{k}] \end{aligned}$$

$$\begin{aligned} rhs_{i,j+1} = {} & \frac{r_1}{2} u_{i-1,j+1}^{k+1} + \frac{r_2}{2} u_{i,j}^{k+1} \\ & + \frac{r_1}{2}\left(u_{i-1,j+1}^{k} + u_{i+1,j+1}^{k}\right) \\ & + \frac{r_2}{2}\left(u_{i,j}^{k} + u_{i,j+2}^{k}\right) \\ & + (2 - r_1 - r_2 - b_1) u_{i,j+1}^{k} \\ & + (2b_k - b_{k-1}) u_{i,j+1}^{0} \\ & - b_k u_{i,j+1}^{1} - \sum_{s=1}^{k-1} (b_{s-1} + 2b_s \\ & + b_{s+1}) u_{i,j+1}^{k-s} + \tau^{\alpha} \Gamma(3-\alpha) f_{i,j+1}^{k}] \end{aligned}$$

Rewrite the above matrix equation as,

$$\begin{pmatrix} u_{i,j}^{k+1} \\ u_{i+1,j}^{k+1} \\ u_{i+1,j+1}^{k+1} \\ u_{i,j+1}^{k+1} \end{pmatrix} = \frac{1}{A} \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ m_2 & m_1 & m_4 & m_3 \\ m_3 & m_4 & m_1 & m_2 \\ m_4 & m_3 & m_2 & m_1 \end{pmatrix} \begin{pmatrix} rhs_{i,j} \\ rhs_{i+1,j} \\ rhs_{i+1,j+1} \\ rhs_{i,j+1} \end{pmatrix} \ldots\ldots\ldots\ldots\ldots (6)$$

Where

$$A = (1 + r_1 + r_2)^4 - \frac{1}{2}(1 + r_1 + r_2)^2({r_1}^2 + {r_2}^2) + \frac{1}{16}({r_1}^2 - {r_2}^2)^2$$

$$m_1 = (1 + r_1 + r_2)\{(1 + r_1 + r_2)^2 - \frac{{r_1}^2}{4} + \frac{{r_2}^2}{4}\}$$

$$m_2 = -\frac{r_1}{2}\{(1 + r_1 + r_2)^2 - \frac{{r_1}^2}{4} + \frac{{r_2}^2}{4}\}$$

$$m_3 = \frac{1}{2}(1 + +r_1 + r_2) r_1 r_2$$

$$m_4 = -\frac{1}{2} r_2 \{(1 + r_1 + r_2)^2 + \frac{{r_1}^2}{4} - \frac{{r_2}^2}{4}\}$$

The scheme described in (6) generate an iterative process on a group of four points over the entire spatial domain. This process continues on a group of four points until certain convergence criterion is achieved. The converged solutions are then utilized as initial guess for the next time level.

Fractional Explicit De-coupled Group (FEDG) Iterative Scheme

To derive the FEDG iterative scheme, similar to FEG method, we apply equation (5) on group of four points will result 4 x 4 system of equations,

$$\begin{pmatrix} k_1 & k_4 & k_3 & k_3 \\ k_4 & k_1 & k_3 & k_3 \\ k_3 & k_3 & k_1 & k_2 \\ k_3 & k_3 & k_2 & k_1 \end{pmatrix} \begin{pmatrix} u_{i,j}^{k+1} \\ u_{i+1,j+1}^{k+1} \\ u_{i+1,j}^{k+1} \\ u_{i,j+1}^{k+1} \end{pmatrix} = \begin{pmatrix} {rhs^*}_{i,j} \\ {rhs^*}_{i+1,j+1} \\ {rhs^*}_{i+1,j} \\ {rhs^*}_{i,j+1} \end{pmatrix}$$

Where

$k_1 = 1 + r_1/2 + r_2/2$, $k_2 = -r_1/4$, $k_3 = 0$ and $k_4 = -r_2/4$

$$\begin{aligned} {rhs^*}_{i,j} = {} & \frac{r_2}{4} u_{i-1,j-1}^{k+1} + \frac{r_1}{4}\left(u_{i-1,j+1}^{k+1} + u_{i+1,j-1}^{k+1}\right) \\ & + \frac{r_1}{4}\left(u_{i-1,j+1}^{k} + u_{i+1,j-1}^{k}\right) \\ & + \frac{r_2}{4}\left(u_{i-1,j-1}^{k} + u_{i+1,j+1}^{k}\right) \\ & + (2 - r_1/2 - r_2/2 - b_1) u_{i,j}^{k} \\ & + (2b_k - b_{k-1}) u_{i,j}^{0} \\ & - b_k u_{i,j}^{1} - \sum_{s=1}^{k-1} (b_{s-1} + 2b_s \\ & + b_{s+1}) u_{i,j}^{k-s} + \tau^{\alpha} \Gamma(3-\alpha) f_{i,j}^{k} \end{aligned}$$

$$rhs^*{}_{i+1,j+1} = \frac{r_2}{4}u_{i,j}^{k+1} + \frac{r_1}{4}\left(u_{i,j+2}^{k+1} + u_{i+2,j}^{k+1}\right) + \frac{r_1}{4}\left(u_{i,j+2}^{k} + u_{i+2,j}^{k}\right) + \frac{r_2}{4}\left(u_{i,j}^{k} + u_{i+2,j+2}^{k}\right) + (2 - r_1/2 - r_2/2 - b_1)u_{i+1,j+1}^{k} + (2b_k - b_{k-1})u_{i+1,j+1}^{0} - b_k u_{i+1,j+1}^{1} - \sum_{s=1}^{k-1}(b_{s-1} + 2b_s + b_{s+1})u_{i+1,j+1}^{k-s} + \tau^{\alpha}\Gamma(3-\alpha)f_{i+1,j+1}^{k}$$

$$rhs^*{}_{i+1,j} = \frac{r_2}{4}u_{i,j-1}^{k+1} + \frac{r_1}{4}\left(u_{i,j+1}^{k+1} + u_{i+2,j-1}^{k+1}\right) + \frac{r_1}{4}\left(u_{i,j+1}^{k} + u_{i+2,j-1}^{k}\right) + \frac{r_2}{4}\left(u_{i,j-1}^{k} + u_{i+2,j+1}^{k}\right) + (2 - r_1/2 - r_2/2 - b_1)u_{i+1,j}^{k} + (2b_k - b_{k-1})u_{i+1,j}^{0} - b_k u_{i+1,j}^{1} - \sum_{s=1}^{k-1}(b_{s-1} + 2b_s + b_{s+1})u_{i+1,j}^{k-s} + \tau^{\alpha}\Gamma(3-\alpha)f_{i+1,j}^{k}$$

$$rhs^*{}_{i,j+1} = \frac{r_2}{4}u_{i-1,j}^{k+1} + \frac{r_1}{4}\left(u_{i-1,j+2}^{k+1} + u_{i+1,j}^{k+1}\right) + \frac{r_1}{4}\left(u_{i-1,j+2}^{k} + u_{i+1,j}^{k}\right) + \frac{r_2}{4}\left(u_{i-1,j}^{k} + u_{i+1,j+2}^{k}\right) + (2 - r_1/2 - r_2/2 - b_1)u_{i,j+1}^{k} + (2b_k - b_{k-1})u_{i,j+1}^{0} - b_k u_{i,j+1}^{1} - \sum_{s=1}^{k-1}(b_{s-1} + 2b_s + b_{s+1})u_{i,j+1}^{k-s} + \tau^{\alpha}\Gamma(3-\alpha)f_{i,j+1}^{k}$$

The above matrix equation of this section can be written as pair of matrix equations,

$$\begin{pmatrix} u_{i,j}^{k+1} \\ u_{i+1,j+1}^{k+1} \end{pmatrix} = \frac{4}{B_1}\begin{pmatrix} n_1 & n_2 \\ n_2 & n_1 \end{pmatrix}\begin{pmatrix} rhs^*{}_{i,j} \\ rhs^*{}_{i+1,j+1} \end{pmatrix} \quad (7)$$

$$\begin{pmatrix} u_{i+1,j}^{k+1} \\ u_{i,j+1}^{k+1} \end{pmatrix} = \frac{4}{B_2}\begin{pmatrix} n^*{}_1 & n^*{}_2 \\ n^*{}_2 & n^*{}_1 \end{pmatrix}\begin{pmatrix} rhs^*{}_{i+1,j} \\ rhs^*{}_{i,j+1} \end{pmatrix} \quad (8)$$

$$B_1 = 4r_1{}^2 + 8r_1(2 + r_2) + 3r_2{}^2 + 16r_2 + 16$$
$$B_2 = 3r_1{}^2 + 8r_1(2 + r_2) + 3r_2{}^2 + 4(r_2 + 2)^2$$
$$n_1 = n^*{}_1 = 2(r_1 + r_2 + 2)$$
$$n_2 = r_2, \qquad n^*{}_2 = r_1$$

The FEDG iterative scheme comprises the two sets of group points represented by the matrix equations (7) and (8). The scheme can be constructed by iterating on either (7) or (8). Suppose the iterations are generated using (7) until a certain criteria is met. Once the convergence is attained the values on the remaining points of the solution domain can be evaluated using FSP formula as described in (4). Similarly, the scheme can be implemented if (8) is chosen for iteration process.

III. Numerical Experiment And Results

Two numerical experiments were performed to test the viability of the proposed methods in solving the two dimensional time-fractional diffusion-wave equation (2.1). The numerical experiments were carried out on a PC with Core 2 Duo 2.8 GHz, 2GB of RAM with Window XP SP3 operating system using Cygwin C and Mathematica 11 software. In both experiments, we assume that the step sizes in both x and y directions are the same. i.e. h = Δx = Δy. Various mesh sizes of 10 , 16 , 22 and 28 were considered for different time steps of 1/10 , 1/16 , 1/22 and 1/28 in example 3.1 and mesh sizes of 10 , 20 , 30 and 40 were considered for different time steps of 1/10 , 1/20 , 1/30 and 1/40 in example 3.2. Gauss Seidel method with relaxation factor ω_e equal to 1 were selected for both examples and for convergence criteria l_∞norm was used with tolerance factor $\varepsilon = 10^{-5}$

TABLE I: Computational Complexity Analysis For Fsp, Frp, Feg And Fedg Methods

Method	Per Iteration	
	(+/-)	(×/÷)
FSP	$(18 + 17(k-1))\lambda^2$	$(10 + (k-1))\lambda^2$
FRP	$((9 + 8.5(k-1))(\lambda^2 + 1)$	$(5 + 0.5(k-1))(\lambda^2 + 1)$
FEG	$(19 + 17(k-1))(\lambda-1)^2 + (18 + 17(k-1))(2\lambda - 1)$	$(16 + (k-1))(\lambda-1)^2 + (10 + (k-1))(2\lambda - 1)$
FEDG	$(9 + 8.5(k-1))(\lambda-1)^2 + (18 + 17(k-1))\lambda$	$(6 + 0.5(k-1))(\lambda-1)^2 + (10 + (k-1))\lambda$

Table 1 summarizes the computational complexity per iteration for FSP, FRP, FEG and FEDG iterative methods in which $k$denote the time level and $\lambda = n - 1$ where n is the mesh size of discretized solution domain whereas Table 2 describes the computational complexity of the four methods after convergence.

TABLE II: COMPUTATIONAL COMPLEXITY ANALYSIS FOR FSP, FRP, FEG AND FEDG METHODS

Method	After Convergence	
	(+/-)	(×/÷)
FSP	-	-
FRP	$((9+8.5(k-1))(\lambda^2-1)$	$(5+0.5(k-1))(\lambda^2-1)$
FEG	-	-
FEDG	$(9+8.5(k-1))(\lambda^2-1)$	$(5+0.5(k-1))(\lambda^2-1)$

Table 3 sums up the total number of arithmetic operations of each iterative method with $Ite.$ indicating the number of iterations.

TABLE III: TOTAL NUMBER OF ARITHMETIC OPERATIONS FOR FSP, FRP, FEG AND FEDG METHODS

Methods	Total Operations
FSP	$(28+18(k-1))\lambda^2 * Ite.$
FRP	$(14+9(k-1))(\lambda^2+1) * Ite.$ $+(14+9(k-1))(\lambda^2-1)$
FEG	$\{((35+18(k-1))(\lambda-1)^2$ $+(28+18(k-1))(2\lambda-1))\} * Ite.$
FEDG	$\{((15+9(k-1))(\lambda-1)^2)+(28+18(k-1))\lambda\} * Ite.$ $+(15+9(k-1))(\lambda^2-1)$

Example 1: Consider the following time-fractional diffusion-wave together with the source term given by the relation [24],

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{5}\sin(x)\sin(y)\,[\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}+t^2]$$

The initial and boundary conditions are given by

$u(x,y,0)=\phi(x,y)=0,\quad u_t\,(x,y,0)=0$

$u(0,y,t)=g_1(y,t)=0,$

$$u(1,y,t)=g_2(1,y,t)=\frac{1}{10}t^2\sin(1)\sin(y)$$

$$u(x,0,t)=g_3(x,t)=0,\quad u(x,1,t)=g_4(x,1,t)=\frac{1}{10}t^2\sin(x)\sin(1)$$

The exact analytical solution is given by

$$u(x,y,t)=\frac{1}{10}t^2\sin(x)\sin(y)$$

Example 2: Consider the following time-fractional diffusion-wave together with the source term given by the relation [24],

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{5}\sin(\pi x)\sin(\pi y)\,[\frac{3t^{3-\alpha}}{\Gamma(4-\alpha)}+\pi^2 t^3]$$

The initial and boundary conditions are given by

$u(x,y,0)=\phi(x,y)=0,\quad u_t\,(x,y,0)=0$

$u(0,y,t)=g_1(y,t)=0,$

$u(1,y,t)=g_2(1,y,t)=0$

$u(x,0,t)=g_3(x,t)=0,\quad u(x,1,t)=g_4(x,1,t)=0$

The exact analytical solution is given by

$$u(x,y,t)=\frac{1}{10}t^3\sin(\pi x)\sin(\pi y)$$

In Table 4-7, the execution timings of FEDG method is only about (30:28104 - 33:82384)% , (86:84919 - 90:196)% and (43:3666 - 46:00035)% of FSP, FRP and FEG methods and total operations of FEDG method is only about (28:85107-30:70429)%, (86:40948 - 95:99756)% and (38:98463 - 39:59843)% of FSP, FRP and FEG methods in example 1 when $\alpha=1.25$. In Table 6-8, execution timings of FEDG method is merely about (29:23623- 32:86743)%, (87:03719-87:97271)% and (42:81756-47:47509)% of FSP, FRP and FEG methods and total operations of FEDG method is merely about (25:83004 - 26:49201)%, (80:27011 - 83:73499)%, (41:94131-43:06748)% of FSP, FRP and FEG methods in example 2 when $\alpha=1.25$. Figures 1 and 2 represent the graphs of FSP, FRP, FEG and FEDG iterative methods of example 1 and 2 in terms of execution times, total operations and number of iterations when $\alpha=1.50$when$\alpha=1.25$respectively.

TABLE IV: COMPARISON BETWEEN FSP, FRP, FEG AND FEDG ITERATIVE METHODS FOR EXAMPLE 1

$\alpha = 1.25$							
Δt	h^{-1}	Method	Time (sec)	*Ite.*	Ave Error	Max Error	Operations
1/10	10	FSP	2.12161	21	2.76949×10^{-3}	5.46536×10^{-3}	323,190
		FRP	0.79561	13	2.81782×10^{-3}	5.56365×10^{-3}	108,870
		FEG	1.56001	15	1.08030×10^{-2}	4.90316×10^{-2}	237,570
		FEDG	0.71761	11	1.06857×10^{-2}	4.74351×10^{-2}	94,074
1/16	16	FSP	16.6297	28	3.02652×10^{-3}	6.44493×10^{-3}	1,877,400
		FRP	6.06844	17	3.02535×10^{-3}	6.45087×10^{-3}	605,834
		FEG	11.9497	20	9.40835×10^{-3}	5.37497×10^{-2}	1,368,440
		FEDG	5.30403	15	9.36129×10^{-3}	5.24839×10^{-2}	541,650
1/22	22	FSP	66.3160	32	3.23760×10^{-3}	7.13215×10^{-3}	5,729,472
		FRP	23.3689	20	3.20596×10^{-3}	7.06987×10^{-3}	1,883,840
		FEG	46.8003	24	8.68431×10^{-3}	5.63772×10^{-2}	4,364,304
		FEDG	20.2957	18	8.64233×10^{-3}	5.53095×10^{-2}	1,712,028
1/28	28	FSP	188.761	36	3.42226×10^{-3}	7.69060×10^{-3}	13,489,416
		FRP	65.4424	22	3.36488×10^{-3}	7.56402×10^{-3}	4,314,516
		FEG	131.400	28	8.25220×10^{-3}	5.81188×10^{-2}	10,624,264
		FEDG	57.1588	21	8.21011×10^{-3}	5.71870×10^{-2}	4,141,830
$\alpha = 1.50$							
Δt	h^{-1}	Method	Time (sec)	*Ite.*	Ave Error	Max Error	Operations
1/10	10	FSP	1.56001	16	4.57450×10^{-3}	9.03902×10^{-3}	246,240
		FRP	0.67081	10	4.65111×10^{-3}	9.19489×10^{-3}	85,500
		FEG	1.24801	11	1.00849×10^{-2}	4.42391×10^{-2}	174,218
		FEDG	0.62400	8	9.99970×10^{-3}	4.21195×10^{-2}	70,512
1/16	16	FSP	11.2009	18	5.39024×10^{-3}	1.15758×10^{-2}	1,206,900
		FRP	4.47723	12	5.40121×10^{-3}	1.16071×10^{-2}	437,464
		FEG	8.58006	13	9.32052×10^{-3}	4.78248×10^{-2}	889,486
		FEDG	4.04043	10	9.28575×10^{-3}	4.58766×10^{-2}	372,300
1/22	22	FSP	42.1359	20	5.99379×10^{-3}	1.33846×10^{-2}	3,580,920
		FRP	15.8653	13	5.97738×10^{-3}	1.33504×10^{-2}	1,255,758
		FEG	31.0286	15	9.11530×10^{-3}	4.98941×10^{-2}	2,727,690
		FEDG	14.2585	11	9.07753×10^{-3}	4.80787×10^{-2}	1,081,146
1/28	28	FSP	113.974	21	6.48216×10^{-3}	1.48200×10^{-2}	7,868,826
		FRP	42.1359	14	6.44448×10^{-3}	1.47320×10^{-2}	2,813,636
		FEG	81.9317	16	9.13409×10^{-3}	5.13371×10^{-2}	6,071,008
		FEDG	37.7990	12	9.07935×10^{-3}	4.96294×10^{-2}	2,447,256

TABLE V: COMPARISON BETWEEN FSP, FRP, FEG AND FEDG ITERATIVE METHODS FOR EXAMPLE 1

$\alpha = 1.75$							
Δt	h^{-1}	Method	Time (sec)	*Ite.*	Ave Error	Max Error	Operations
1/10	10	FSP	1.20121	12	6.78819×10^{-3}	1.36466×10^{-2}	184,680
		FRP	0.56160	8	6.88497×10^{-3}	1.39692×10^{-2}	69,920
		FEG	1.07641	8	1.04051×10^{-2}	3.95971×10^{-2}	126,704
		FEDG	0.54600	6	1.03627×10^{-2}	3.71935×10^{-2}	54,804
1/16	16	FSP	7.78445	13	8.27462×10^{-3}	1.82886×10^{-2}	871,650
		FRP	3.27602	8	8.30315×10^{-3}	1.83985×10^{-2}	302,768
		FEG	6.67684	9	1.03806×10^{-2}	4.14124×10^{-2}	615,798
		FEDG	3.10722	7	1.03602×10^{-2}	3.89262×10^{-2}	270,690
1/22	22	FSP	27.5186	13	9.29067×10^{-3}	2.14903×10^{-2}	2,327,598
		FRP	11.4193	9	9.29111×10^{-3}	2.15074×10^{-2}	896,854
		FEG	22.6357	10	1.06511×10^{-2}	4.23311×10^{-2}	1,818,460
		FEDG	11.1073	7	1.06248×10^{-2}	3.98208×10^{-2}	720,642
1/28	28	FSP	71.8229	13	1.00521×10^{-2}	2.38828×10^{-2}	4,871,178
		FRP	29.2190	9	1.00380×10^{-2}	2.38649×10^{-2}	1,875,586
		FEG	57.4240	10	1.10068×10^{-2}	4.29659×10^{-2}	3,794,380
		FEDG	28.0334	8	1.09631×10^{-2}	4.04566×10^{-2}	1,694,112

TABLE VI: COMPARISON BETWEEN FSP, FRP, FEG AND FEDG ITERATIVE METHODS FOR EXAMPLE 2

$\alpha = 1.25$							
Δt	h^{-1}	Method	Time (sec)	*Ite.*	Ave Error	Max Error	Operations
1/10	10	FSP	2.23081	25	8.64546×10^{-3}	1.75656×10^{-2}	384,750
		FRP	0.84241	15	7.98274×10^{-3}	1.61696×10^{-2}	124,450
		FEG	1.54441	15	8.59709×10^{-3}	1.74673×10^{-2}	237,570
		FEDG	0.73321	12	7.93132×10^{-3}	1.60709×10^{-2}	101,928
1/20	20	FSP	49.8891	39	1.00902×10^{-2}	2.25594×10^{-2}	5,209,230
		FRP	17.2069	23	9.91067×10^{-3}	2.21552×10^{-2}	1,606,910
		FEG	33.4154	23	1.00410×10^{-2}	2.24517×10^{-2}	3,124,274
		FEDG	14.9917	19	9.89806×10^{-3}	2.21308×10^{-2}	1,345,546
1/30	30	FSP	291.659	47	1.10372×10^{-2}	2.54926×10^{-2}	21,739,850
		FRP	97.0638	29	1.08793×10^{-2}	2.51271×10^{-2}	6,945,950
		FEG	192.256	28	1.09318×10^{-2}	2.52492×10^{-2}	13,105,064
		FEDG	85.2701	23	1.08527×10^{-2}	2.50667×10^{-2}	5,575,522
1/40	40	FSP	972.869	52	1.18020×10^{-2}	2.77097×10^{-2}	57,737,160
		FRP	329.193	32	1.15925×10^{-2}	2.72174×10^{-2}	18,331,760
		FEG	676.358	32	1.16145×10^{-2}	2.72704×10^{-2}	35,854,016
		FEDG	289.600	26	1.15446×10^{-2}	2.71078×10^{-2}	15,037,644
$\alpha = 1.50$							
Δt	h^{-1}	Method	Time (sec)	*Ite.*	Ave Error	Max Error	Operations
1/10	10	FSP	1.66921	18	1.35198×10^{-2}	2.74697×10^{-2}	277,020
		FRP	0.68640	11	1.30378×10^{-2}	2.64471×10^{-2}	93,290
		FEG	1.32601	12	1.34192×10^{-2}	2.72673×10^{-2}	190,056
		FEDG	0.62400	9	1.29331×10^{-2}	2.62436×10^{-2}	78,366
1/20	20	FSP	30.5606	23	1.71539×10^{-2}	3.83538×10^{-2}	3,072,110
		FRP	11.2945	15	1.70329×10^{-2}	3.80816×10^{-2}	1,071,150
		FEG	21.0601	14	1.71062×10^{-2}	3.82493×10^{-2}	1,901,732
		FEDG	9.90606	12	1.70187×10^{-2}	3.80519×10^{-2}	874,488
1/30	30	FSP	157.062	25	1.93724×10^{-2}	4.47459×10^{-2}	11,563,750
		FRP	57.2212	16	1.92505×10^{-2}	4.44644×10^{-2}	3,935,800
		FEG	109.513	16	1.92682×10^{-2}	4.45053×10^{-2}	7,488,608
		FEDG	50.5131	13	1.92208×10^{-2}	4.43965×10^{-2}	3,252,182
1/40	40	FSP	486.458	26	2.09998×10^{-2}	4.93060×10^{-2}	28,868,580
		FRP	179.354	16	2.08277×10^{-2}	4.89022×10^{-2}	9,443,280
		FEG	343.779	16	2.08251×10^{-2}	4.88973×10^{-2}	17,927,008
		FEDG	160.400	13	2.07757×10^{-2}	4.87821×10^{-2}	7,796,982

TABLE VII: COMPARISON BETWEEN FSP, FRP, FEG AND FEDG ITERATIVE METHODS FOR EXAMPLE 2

$\alpha = 1.75$							
Δt	h^{-1}	Method	Time (sec)	*Ite.*	Ave Error	Max Error	Operations
1/10	10	FSP	1.26361	13	1.88837×10^{-2}	3.83680×10^{-2}	200,070
		FRP	0.56160	9	1.85557×10^{-2}	3.76675×10^{-2}	77,710
		FEG	1.06081	9	1.87138×10^{-2}	3.80243×10^{-2}	142,542
		FEDG	0.57720	7	1.83796×10^{-2}	3.73127×10^{-2}	62,658
1/20	20	FSP	18.8761	14	2.46013×10^{-2}	5.50069×10^{-2}	1,869,980
		FRP	7.92485	9	2.45285×10^{-2}	5.48430×10^{-2}	669,330
		FEG	14.3833	9	2.45249×10^{-2}	5.48376×10^{-2}	1,222,542
		FEDG	7.64405	8	2.44649×10^{-2}	5.47020×10^{-2}	605,312
1/30	30	FSP	90.8862	14	2.75876×10^{-2}	6.37226×10^{-2}	6,475,700
		FRP	36.7226	9	2.75107×10^{-2}	6.35452×10^{-2}	2,314,950
		FEG	68.8432	9	2.75056×10^{-2}	6.35326×10^{-2}	4,212,342
		FEDG	35.2562	8	2.74833×10^{-2}	6.34824×10^{-2}	2,090,512
1/40	40	FSP	268.571	13	2.95177×10^{-2}	6.93065×10^{-2}	14,434,290
		FRP	108.452	9	2.94089×10^{-2}	6.90514×10^{-2}	5,554,570
		FEG	208.090	9	2.93914×10^{-2}	6.90107×10^{-2}	10,083,942
		FEDG	101.416	7	2.93673×10^{-2}	6.89547×10^{-2}	4,455,138

In both figures, one can easily observed that FEDG iterative method requires the least number of total numbers of arithmetic operations and CPU timings as compared to other three methods state.

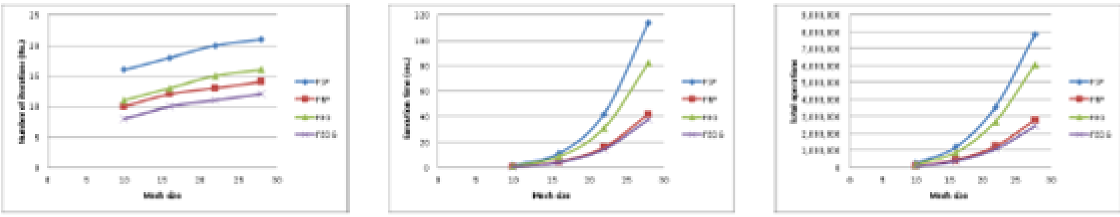

Figure 1: Graph of FSP, FRP, FEG and FEDG when α =1.50 for Example 1

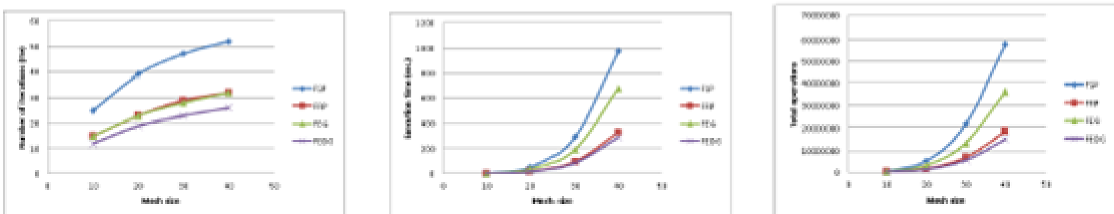

Figure 2: Graph of FSP, FRP, FEG and FEDG when α =1.25 for Example 2

IV. CONCLUSION

In this study, we have developed two new groups iterative methods derived from the fractional standard five points and fractional rotated five point schemes in solving the two dimensional second order diffusion wave equation. The fractional rotated five point scheme is derived from the fractional standard five point scheme by rotating the clockwise at an angle of 45° from the standard mesh. Consequently, FEDG iterative method is based on the involvement of rotation. Our findings indicate that FEDG requires the least computing efforts in terms of computational complexity and least CPU execution times among the other methods tested.

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