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ANALYTICAL SOLUTIONS OF NONLINEAR SCHRODINGER EQUATIONS USING MULTISTEP MODIFIED REDUCED DIFFERENTIAL TRANSFORM METHOD

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Abstract: This paper aims to propose and implement the Multistep Modified Reduced Differential Transform Method (MMRDTM) to find solution of Nonlinear Schrodinger Equations (NLSEs). Through the proposed technique, we replaced the nonlinear term in the NLSEs by the corresponding Adomian polynomials prior applying the multistep approach. Thus, we can obtain solutions for the NLSEs in easier way with less complexity. In addition, the solutions can be approximated more accurately over a longer time frame. We considered several NLSEs and illustrate the features of these solutions in the form of graphs in order to show the power and accuracy of the MMRDTM.

Keywords: Adomian polynomials, multistep approach, Reduced Differential Transform Method, nonlinear Schrodinger equations

I. INTRODUCTION

In the study of nonlinear physical phenomena, it is an important problem to investigate traveling wave for solutions of Nonlinear Evolution Equations. This is also vital to explain the certain natural phenomena in science and engineering. The analytical nonlinear solutions techniques are important in solving nonlinear problems because the solutions obtained can be further investigated in great detail to yield important information. Recently, some vital enhancement for finding analytical solutions of Nonlinear Schrodinger Equations (NLSEs) can be found in literatures [1].

The NLSE is an instance of a nonlinear mathematical equation that explains nonlinear systems appearing in various nonlinear phenomena [2]. In fact, a great deal of research has been carried out to find efficient analytical methods for solving the NLSEs. For obtaining the solutions of Schrodinger equations, Sadighi & Ganji [3] studied Adomian decomposition method (ADM) and homotopy perturbation method (HPM).Previously, Biazar & Ghazvini[4] proposed He's HPM to solve cubic Schrodinger equation. Furthermore, Bratsos et al. [5] proposed a new discrete ADM to solve discrete NLSEs. On the other hand, Wazwaz [6] applied the variational iteration method to solve the linear Schrodinger equations and as well as NLSEs. For the same purpose, Ravi Kanth & Aruna[7] proposed two-dimensional differential transform method (DTM). Recently in 2016, Taghizadeh & Noori[8] used the reduced differential transform method (RDTM) to solve the NLSE with cubic nonlinearity.

Currently, several partial differential equations, ordinary differential equations and delay differential equations have been solved by using DTM and RDTM [9-13]. However, Ray [14] proposed a modification on the fractional RDTM and implemented it to find solutions of fractional Korteweg de Vries (KdV) equations. Through this approach, the modification involved the replacement of the nonlinear term by the corresponding Adomian polynomials. Consequently, the nonlinear initial value problem now can obtain solutions in easier way with less complexity. From the results, the proposed method was proven to be more effective and simpler for obtaining the approximation to the solutions of fractional KdV equations. Furthermore, El-Zahar [15]presented adaptive multistep DTM to solve singular perturbation initial-value problems. The solutions can be easily compute over a sequence of sub-intervals with variable-length by using piecewise convergent series.

In this study, we propose a multistep modified RDTM to solve NLSEs. This paper aims to combine the idea of the modification in Ray [14] and multistep approach in El-Zahar [15]and apply it to obtain analytical solutions of NLSEs. With this modification, our method would solve NLSEs with different nonlinearity and then apply multistep approach to achieve high accuracy of the solutions for wide time frame.

II. The Development Of Multistep Modified Reduced Differential Transform Method (MMRDTM)

For notation purpose, the original functions will be denoted using lowercase letter such as the letter u in the function $u(x,t)$, while the transformed functions will be denoted using uppercase letter such as the letter U in the function $U_k(x)$. Basically, the differential transformation of the function $u(x,t) = f(x)g(t)$ is given by Keskin & Oturanç [16],

$$u(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{j=0}^{\infty} U_m(x)t^m$$

where $U_m(x)$ is known as the function of $u(x,t)$. Some fundamental properties of RDTM are given as follows:

Definition 1. For an analytically and continuously differential function $u(x,t)$ with respect to time t and space variable x, the differential transformation of $u(x,t)$ is defined by

$$U_m(x) = \frac{1}{m!}\left[\frac{\partial^m}{\partial t^m} u(x,t)\right]_{t=0} \quad (2.1)$$

where function $U_m(x)$is the transformed function.

Definition 2. The inverse transform of $U_m(x)$is given by

$$u(x,t) = \sum_{j=0}^{\infty} U_m(x)t^m. \quad (2.2)$$

Then by combining equations (2.1) and (2.2), we can write the equation as following

$$u(x,t) = \sum_{m=0}^{\infty} \frac{1}{m!}\left[\frac{\partial^m}{\partial t^m} u(x,t)\right]_{t=0} t^m. \quad (2.3)$$

In order to represent the core properties of the RDTM, nonlinear partial differential

$$Du(x,t) + Pu(x,t) + Qu(x,t) = h(x,t), \quad (2.4)$$

is considered with initial condition$u(x,\ 0) = f(x)$.

Here, $D = \frac{\partial}{\partial t}$ and P is the remaining part of linear operator. The nonlinear and inhomogeneous terms are represented as $Qu(x,t)$ and$h(x,t)$ respectively.

Based on the MMRDTM, the iteration formula can be formed as follows:

$$(m+1)U_{m+1}(x) = H_m(x) - PU_m(x) - QU_m(x). \quad (2.5)$$

The functions $Du(x,t), Pu(x,t), Qu(x,t)$ and $h(x,t)$ are transformed and then represented as $U_m(x), PU_m(x), QU_m(x)$ and $H_m(x)$respectively.We have

$$U_0(x) = f(x), \quad (2.6)$$

From the initial condition. Referring to Ray [14], the nonlinear term is denoted as

$$Qu(x,t) = \sum_{n=0}^{\infty} A_n\big(U_0(x), U_1(x), \dots, U_n(x)\big).$$

By combining equation (2.6) into equation (2.5) and through iterative calculation, the $U_m(x)$ values can be obtained. Furthermore, the set of values $\{U_m(x)\}_{m=0}^{n}$ of the inverse transformation yields the following approximate solution,

$$u(x,t) = \sum_{m=0}^{M} U_m(x)t^m, \qquad t \in [0,T].$$

For $i = 1,2,\dots,N$, the interval $[0,T]$ is divided into N subintervals $[t_{i-1}, t_i]$ by equalizing the step size of $s = \frac{T}{N}$ and nodes $t_i = is$ are used. MMRDTM is computed according to the following steps. Firstly, the RDTM is applied to the initial value problem of interval $[0, t_1]$. From there, the approximate result

$$u_1(x,t) = \sum_{m=0}^{M} U_{m,1}(x)t^m, \qquad t \in [0, t_1]$$

is obtained using the initial conditions $u(x,0) = f_0(x)$, $u_1(x,0) = f_1(x)$.

For $i \geq 2$, the initial conditions$u_i(x,t_{i-1}) = u_{i-1}(x,t_{i-1}), (\partial/\partial t)u_i(x,t_{i-1}) = (\partial/\partial t)u_{i-1}(x,t_{i-1})$is used at each subinterval $[t_{i-1}, t_i]$, and the MRDTM is applied to the initial value problem on $[t_{i-1}, t_i]$, where t_0 is substituted by t_{i-1}. For $i = 1,2,\dots,K$, the process is continued and repeatedly performed to get approximate solutions $u_i(x,t)$ in sequence form that is

$$u_i(x,t) = \sum_{m=0}^{M} U_{m,i}(x)(t-t_{i-1})^m, \qquad t \in [t_{i-1}, t_i].$$

Finally, the MMRDTM proposes the following solutions:

$$u(x,t) = \begin{cases} u_1(x,t), for\ t \in [0,t_1] \\ u_2(x,t), for\ t \in [t_1,t_2] \\ \vdots \\ u_N(x,t), for\ t \in [t_{N-1},t_N]. \end{cases}$$

Thus, we found that the obtained algorithm of MMRDTM is simple with better computational performance for all values of s. If the step size $s = T$, we can easily observe that the MMRDTM reduces to the MRDTM .

III. Application Of MMRDTM To Solve The Nonlinear Schrodinger Equations

Consider the general NLSE of the form

$$iu_t + u_{xx} + \gamma|u|^2 = 0, \qquad i = \sqrt{-1}$$

(3.1)

with initial condition$u(x,0) = f(x)$.

In equation (3.1), γ is a constant and $u(x,t)$ is a complex function. This type of equation has been widely utilized particularly to study nonlinear waves. Using basic properties of MMRDTM and then applying MMRDTM to equation (3.1), we obtain

$$U_{m+1,i}(x) = \left(\frac{I}{m+1}\right)\left(\frac{\partial^2}{\partial x^2}\left(U_{m,i}(x)\right) + \gamma\sum_{m=0}^{n} A_{m,i}\right) \qquad (3.2)$$

with transformed initial condition

$$U_0(x) = f(x).$$

The nonlinear term is given as follows

$$Qu(x,t) = \sum_{n=0}^{\infty} A_n(U_0(x), U_1(x), \dots, U_n(x)).$$

By substituting equation (3.3) into equation (3.2) and applying iterative calculation, we may obtain the $U_m(x)$ values. Furthermore, the set of values $\{U_m(x)\}_{m=0}^{n}$ of the inverse transformation yields the next approximate solution

$$u(x,t) = \sum_{m=0}^{M} U_m(x)t^m, \qquad t \in [0,T].$$

For $i = 1,2,\dots,N$, the interval $[0,T]$ is divided into N subintervals $[t_{i-1}, t_i]$ by equalizing the step size of $s = \frac{T}{N}$ and nodes $t_i = is$ are used. The computation is started by applying RDTM to the initial value problem of interval $[0, t_1]$. From there, the approximate result

$$u_1(x,t) = \sum_{m=0}^{M} U_{m,1}(x)t^m, \qquad t \in [0,t_1]$$

is obtained using the initial conditions $u(x,0) = f_0(x)$, $u_1(x,0) = f_1(x)$. For $i \geq 2$, the initial conditions $u_i(x,t_{i-1}) = u_{i-1}(x,t_{i-1}), (\partial/\partial t)u_i(x,t_{i-1}) = (\partial/\partial t)u_{i-1}(x,t_{i-1})$ are used at each subinterval $[t_{i-1}, t_i]$, and the MRDTM is applied to the initial value problem of interval $[t_{i-1}, t_i]$. Here, the time t_0 is substituted by t_{i-1}. For $i = 1,2,\dots,N$, the process is continuously and repeatedly performed to create a sequence of approximate solutions $u_i(x,t)$, for the solution $u(x,t)$ such as

$$u_i(x,t) = \sum_{m=0}^{M} U_{m,i}(x)(t-t_{i-1})^m, \qquad t \in [t_{i-1}, t_i].$$

The MMRDTM assumes the solution as follows:

$$u(x,t) = \begin{cases} u_1(x,t), \text{for}\ t \in [0,t_1] \\ u_2(x,t), \text{for}\ t \in [t_1,t_2] \\ \vdots \\ u_N(x,t), \text{for}\ t \in [t_{N-1},t_N]. \end{cases}$$

IV. Numerical Results And Discussions

To demonstrate the accuracy of the MMRDTM and it's advantageous for solving NLSE, consider the following two numerical examples:

*Example 4.1*One-dimensional cubic NLSE is considered by Ravi Kanth & Aruna [7] of the form

$$iu_t + u_{xx} - 2|u|^2u = 0 \qquad (4.1)$$

with initial condition$u(x,0) = e^{ix}$. The exact solution is $e^{i(x-3t)}$.

Using basic properties of MMRDTM and then applying MMRDTM to Eq. (4.1), we obtain

$$U_{m+1,i}(x) = \left(\frac{I}{m+1}\right)\left(\frac{\partial^2}{\partial x^2}\left(U_{m,i}(x)\right) - 2\sum_{m=0}^{n} A_{m,i}\right). \qquad (4.2)$$

(3.3)

We write the transformed initial condition as $U_0(x) = e^{ix}$. For $i = 1,2,\dots,10$, the interval [0,1] is divided into 10 subintervals $[t_{i-1}, t_i]$of equal step sizeby using the nodes $t_i = is$.Then, the MRDTM is applied to the initial value problem of interval$[t_{i-1}, t_i]$, where t_0 is substituted by t_{i-1}. The process is continuously and repeatedly performed to get approximate solutions $u_i(x,t), i = 1,2,\dots,10$, for the solution $u(x,t)$, such as

$$u_i(x,t) = \sum_{m=0}^{M} U_{m,i}(x)(t-t_{i-1})^m, \qquad t \in [t_{i-1}, t_i].$$

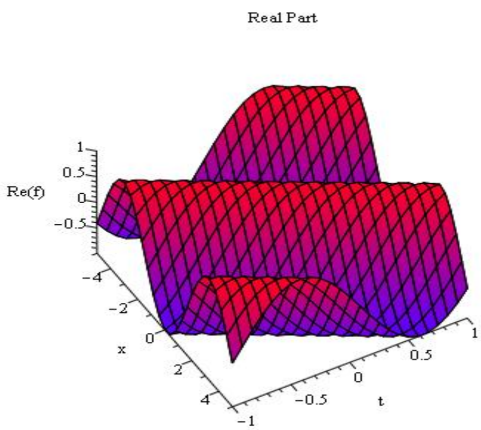

Figure 4.1: (a) Exact solution n

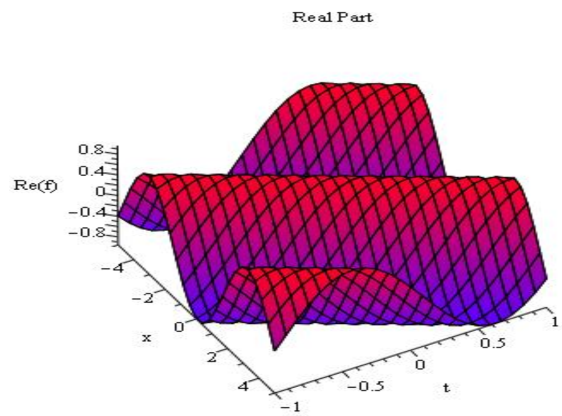

Figure 4.2: (a) MMRDTM

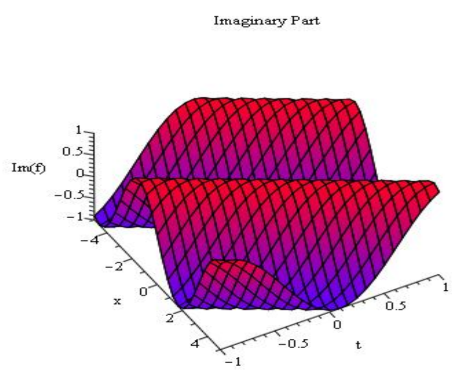

Figure 4.1: (b) Exact solution

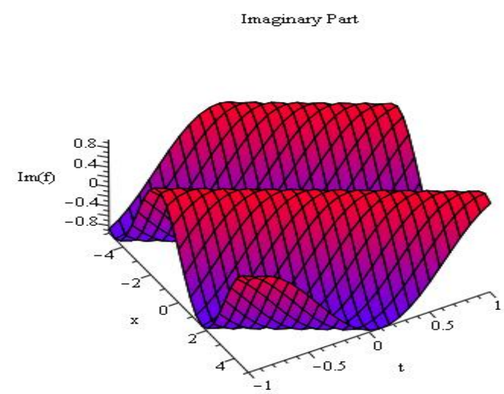

Figure 4.2: (b) MMRDTM

TABLE IV.I: COMPARISON ERROR RESULTS OF MMRDTM AND RDTM APPROXIMATE SOLUTIONS

T	***Exact Solution***	***Absolute Error MMRDTM***	***Absolute Error RDTM***
0.1	$0.9800665778 + 0.1986693308 * I$	$1.00000000\ 10^{-10}$	$4.337211085\ 10^{-8}$
0.2	$0.9210609940 + 0.3894183423 * I$	$2.777516877\ 10^{-8}$	$5.542145993\ 10^{-6}$
0.3	$0.8253356149 + 0.5646424734 * I$	$1.274754878\ 10^{-8}$	$9.443482842\ 10^{-5}$
0.4	$0.6967067093 + 0.7173560909 * I$	$5.370009311\ 10^{-8}$	$7.047729930\ 10^{-4}$
0.5	$0.5403023059 + 0.8414709848 * I$	$1.002241987\ 10^{-7}$	$3.344240614\ 10^{-3}$
0.6	$0.3623577545 + 0.9320390860 * I$	$1.509450231\ 10^{-7}$	$1.191194101\ 10^{-2}$
0.7	$0.1699671429 + 0.9854497300 * I$	$2.063280155\ 10^{-7}$	$3.479927350\ 10^{-2}$
0.8	$-0.2919952230e - 1 + 0.9995736030 * I$	$2.6599024421\ 10^{-7}$	$8.790720689\ 10^{-2}$
0.9	$-0.2272020947 + 0.9738476309 * I$	$3.302733716\ 10^{-7}$	$1.986832756\ 10^{-1}$
1.0	$-0.4161468365 + 0.9092974268 * I$	$3.988355049\ 10^{-7}$	$4.112448302\ 10^{-1}$

The exact solutions are shown in Fig. 4.1(a) and Fig. 4.1(b) while the results of approximate solution MMRDTM for $t \in$ [-1,1] and $x \in$ [-5,5] which involve real part and imaginary part are shown in Fig. 4.2(a) and Fig. 4.2(b). Therefore, it reveals that the multistep approximate solutions for this type of NLSE are extremely close to the exact solutions. The performance error analyses obtained by MMRDTM are summarized in Table 4.1.

*Example 4.2*Ravi Kanth & Aruna [7] considered one-dimensional NLSE with trapping potential in the form

$$iu_t = -\frac{1}{2}u_{xx} + u\,cos^2(x) + |u|^2u = 0, t \geq 0 \qquad (4.3)$$

with initial condition $u(x,0) = e^{ix}$. The exact solution is $\sin x\, e^{(-\frac{3it}{2})}$.

Using basic properties of MMRDTM and then applying MMRDTM to equation (4.3), we can obtain

$$U_{m+1,i}(x) = \left(\frac{I}{m+1}\right)\left(\frac{1}{2}.\frac{\partial^2}{\partial x^2}\left(U_{m,i}(x)\right) - U_m(x).(\cos(x))^2 - \sum_{m=0}^{n} A_{m,i}\right). \qquad (4.4)$$

We write transformed initial condition as $U_0(x) = \sin x$.

Fig. 4.3(a) and Fig. 4.3(b) show the exact solutions. The results of approximate solution MMRDTM for $t \in$ [-1,1] and $x \in$ [-5,5] which involve real part and imaginary part are shown in Fig. 4.4(a) and Fig. 4.4(b). Therefore, it is obvious that the multistep approximate solutions for this type of NLSE are very close to the exact solutions.

The performance error analyses obtained by MMRDTM are summarized in Table 4.2.

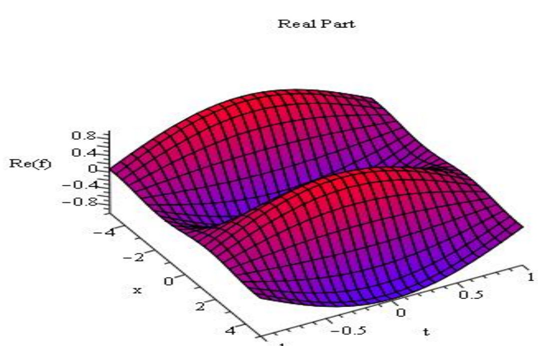

Figure 4.3: (a) Exact solution

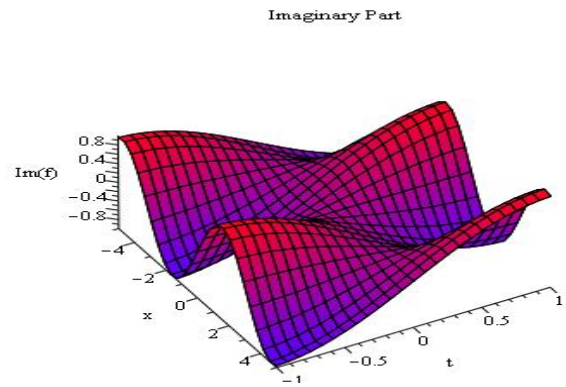

Figure 4.3: (b) Exact solution

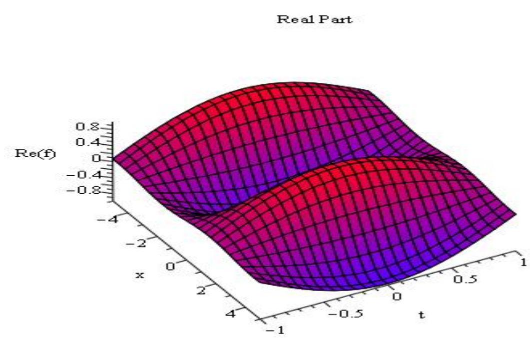

Figure 4.4: (a) MMRDTM

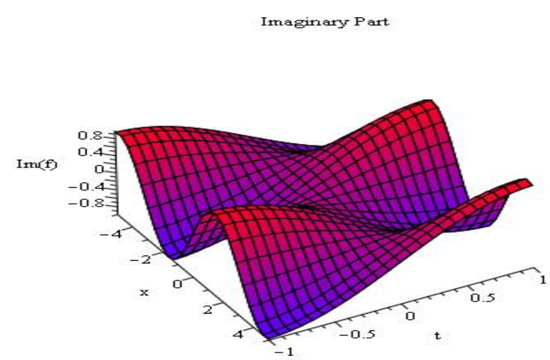

Figure 4.4: (b) MMRDTM

TABLE IV.II: COMPARISON ERROR RESULTS OF MMRDTM AND RDTM APPROXIMATE SOLUTIONS

T	*Exact Solution*	*Absolute Error MMRDTM*	*Absolute Error RDTM*
0.1	$0.1964384884 - 0.2968877378e - 1 * I$	$6.000000000\ 10^{-11}$	$6.000000000\ 10^{-11}$
0.2	$0.3720255519 - 0.1150809890 * I$	$2.000000000\ 10^{-10}$.	$1.691064753\ 10^{-8}$
0.3	$0.5084306791 - 0.2456000150 * I$	$6.708203932\ 10^{-10}$	$4.180610123\ 10^{-7}$
0.4	$0.5920595304 - 0.4050497175 * I$	$1.118033989\ 10^{-9}$	$3.975654161\ 10^{-6}$
0.5	$0.6156949531 - 0.5735792387 * I$	$1.640121947\ 10^{-9}$	$2.221027253\ 10^{-5}$
0.6	$0.5793647867 - 0.7300912969 * I$	$2.193171220\ 10^{-9}$	$8.801702383\ 10^{-5}$
0.7	$0.4903312548 - 0.8548019835 * I$	$2.745906044\ 10^{-9}$	$2.732912098\ 10^{-4}$
0.8	$0.3622032462 - 0.9316416673 * I$	$3.264965543\ 10^{-9}$	$7.044724325\ 10^{-4}$
0.9	$0.2132791434 - 0.9502058804 * I$	$3.605551275\ 10^{-9}$	$1.561726726\ 10^{-3}$
1.0	$0.6432115546e - 1 - 0.9070196246 * I$	$3.716517187\ 10^{-9}$	$3.040909211\ 10^{-3}$

V. Conclusions

In this study, a new approximate analytical method, so-called MMRDTM, is proposed and applied to handle one-dimensional nonlinear Schrodinger equations. In this new approach, the modification involved the replacement of nonlinear term by its Adomian polynomials and a multistep approach was adapted. As the results along with the graphical representations showed, the approximate solutions of NLSE with high accuracy were obtained. In conclusion, in obtaining analytic approximate solution for these types of equations, we can say that the MMRDTM is very powerful, reliable and efficient. All computations in this paper had been carried out by using Maple 13.

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