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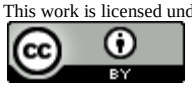

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BERNSTEIN'S APPROXIMATION OF GENERALIZED ABEL'S INTEGRAL EQUATION WITH APPLICATION IN TOMOGRAPHY

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Abstract: This paper presents the Bernstein and hybrid Bernstein approximations to solve the generalized Abel's integral equations (GAIEs) via collocation approach. Bernstein polynomial and hybrid Bernstein functions are used in the approximation of GAIEs solutions and convergence analysis are presented in detail. To show the validity of the proposed methods, numerical examples are considered and an application is shown through Abel inversion in tomography.

Keywords: Abel's equation; Bernstein polynomial; Block-Pulse function; Collocation method; Hybrid Bernstein Block- Pulse function.

I. INTRODUCTION

Integral equations with weakly singular kernel create the base for many evolutionary models in the field of science and engineering [17]. Similar equations were studied by Abel and Volterra under different conditions and such integral equations under regularity condition were named as the Volterra integral equations.

The GAIEs is given by,

$$a(\gamma)\int_r^\gamma \varphi(\delta)k(\gamma,\delta)d\delta + b(\gamma)\int_\gamma^s \varphi(\delta)k(\gamma,\delta)d\delta = \zeta(\gamma),\ \gamma \in (r,s) \quad (1)$$

where the function $\varphi(\gamma)$ is unknown which we have to approximate and $a(\gamma), b(\gamma), \zeta(\gamma)$ and $k(\gamma,\delta) = \frac{1}{(\gamma-\delta)^\mu}, 0 < \mu < 1$, is the power kernel of convolution type.

For, $a(\gamma) = 1, b(\gamma) = 0$, the above equation is converted into the standard form of Abel's integral equations defined as,

$$\int_0^\gamma \varphi(\delta)k(\gamma,\delta)d\delta = \zeta(\gamma), \quad (2)$$

which is a form of first kind Volterra integral equation.

In 1993, Chakrabarty and George [5] derived a formula to obtain the analytical solution of GAIEs using fractional calculus. Further, a theoretic method to obtain the closed form of the exact solution is discussed in [6]. Some numerical methods to approximate the solution of generalized Abel's integral equations can be found in [7,10,15]. Recently, many authors have used the Bernstein's polynomials to solve the system of differential equations [2, 12], integral equations [9,11], isoperimetric variational problems [13], the problems of astronomy [12,16].

This article uses Bernstein's polynomials and hybrid Bernstein Block-Pulse functions (HBBPF) [3, 4] coupled with the collocation approach to develop the numerical schemes for the integral equation defined by Eq. (1). This approach is beneficial and computationally efficient as we obtain good numerical approximations by using only few basis functions.
Rest of the paper is organized as follows: Section 2 explains the basic of Bernstein polynomials. Section 3 presents the methods which are used to approximate the solution of Eq. (1). Section 4 presents the convergence analysis of the proposed methods. Numerical experiments are performed in section 5. An application of Abel inversion is added in section 6 and last section concludes the paper.

II. BASIC DEFINITIONS AND RESULTS

Definitions 1 [8]. Bernstein basis polynomial over $[0,1]$ are given by,

$$\mathcal{B}_{n,k}(\gamma) = \binom{n}{k}\gamma^k(1-\gamma)^{n-k}, \quad k = 0, \dots, n. \tag{3}$$

For our convenience, we define $\mathcal{B}_{n,k}(\gamma) = 0, k < 0 \; or \; k > n$.

Theorem 1 [9]. For any continuous function φ defined on the interval $[0,1]$ the Bernstein polynomial approximation of φ defined by

$$\mathcal{B}_n\big(\varphi(\gamma)\big) = \sum_{k=0}^{n} \varphi(k/n)\, \mathcal{B}_{n,k}(\gamma), \tag{4}$$

satisfies, $\lim_{n\to\infty} \mathcal{B}_n\big(\varphi(\gamma)\big) = \varphi(\gamma).$ (5)

Thus Bernstein polynomial is a way to prove the Weierstrass approximation states that for any function which is continuous on a bounded interval $[0,1]$ can be approximated uniformly by a sequence of polynomials. This theorem implies that for any continuous function φ and for any $\epsilon > 0$, there exists an n such that,

$$\big\|\mathcal{B}_n\big(\varphi(\gamma)\big) - \varphi\| < \epsilon, \tag{6}$$

holds. It has been shown that n satisfies

$$n > \frac{\|\varphi\|}{\rho^2\epsilon}, \tag{7}$$

where $\|*\|$sup norm on $[0,1]$ and since φ is uniformly continuous on [0,1], we for $\gamma, \delta \in [0,1]$ such that $|\gamma - \delta| < \rho$, and $|\varphi(\gamma) - \varphi(\delta)| < \epsilon$ [9].
and the error bound,

$$|\mathcal{B}_n\big(\varphi(\gamma)\big) - \varphi| \le \frac{1}{2n}\gamma(1-\gamma)\|\varphi''\| \le \frac{1}{8n} \| \varphi'' \|, \tag{8}$$

from which it follows that,

$$\lim_{n\to\infty} n(\, B_n\big(\varphi(\gamma)\big) - \varphi(\gamma)) = \frac{1}{2}\gamma(1-\gamma)\|\varphi''\|. \tag{9}$$

The Eq. () establishes the rate of convergence $\frac{1}{n}$ for
$\varphi \in C^2[0,1]$ [9].

Definitions 3 [8]. The HBBPF $\mathscr{b}_{m,k}(\gamma)$ on the interval $[a, b]$ are defined as

$$\mathscr{b}_{m,k}(\gamma) = \begin{cases} \binom{n}{k} \frac{(\gamma - l_{m-1})^k (l_m - \gamma)^{n-k}}{(l_m - l_{m-1})^n}, & \gamma \in [l_m, l_{m-1}] \\ 0 & \gamma \notin [l_m, l_{m-1}] \end{cases}, \tag{10}$$

for $m = 1,2, \dots, M$ and $k = 0,1, \dots, n$,

where $l_m = \frac{(M-m)a + mb}{M}$, $\quad m = 1,2, \dots, M.$ (11)

III. OUTLINE OF METHOD

This section describes the methods to approximate the solution of Eq. (1).

3.1 Bernstein polynomials method (S1): Any unknown function φ can be written in the linear combination of Bernstein basis polynomials as

$$\mathcal{B}_n\left(\varphi_n(\gamma)\right) = \sum_{k=0}^{n} \varphi_n(k/n)\, \mathcal{B}_{n,k}(\gamma), \qquad t \in [0,1]. \tag{12}$$

Here $\varphi_n(k/n)$ are known as Bernstein coefficients which we have to calculate.

To approximate the solution of Eq. (1), we consider the following form:

$$\left(a(\gamma) \int_0^{\gamma} \sum_{k=0}^{n} \varphi_n\left(\frac{k}{n}\right) \mathcal{B}_{n,k}(\gamma) k(\gamma,\delta) d\delta + b(\gamma) \int_{\gamma}^{1} \sum_{k=0}^{n} \varphi_n\left(\frac{k}{n}\right) \mathcal{B}_{n,k}(\gamma) k(\delta,\gamma) d\delta\right) = \zeta(\gamma), \tag{13}$$

In order to find the unknowns, $\varphi_n\left(\frac{k}{n}\right), k = 0,1,\dots,n$, we replace γ by $\gamma_k \in [0,1],\ k = 0,1,\dots,n$ and convert Eq. (13) into a system of algebraic equations.

3.2 Hybrid Bernstein block-pulse functions method (S2): This sectiondescribe the use of HBBPF approximation of the solution of Eq. (1).

Any function $f \in L^2[0,1)$ can be written as

$$f(\gamma) = \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} p_{m,k}\, \mathscr{b}_{m,k}(\gamma). \tag{14}$$

After truncating it to some finite values n and M, we obtain

$$f(\gamma) = \sum_{m=1}^{M} \sum_{k=0}^{n} p_{m,k}\, \mathscr{b}_{m,k}(\gamma) = P^T H(\gamma), \tag{15}$$

Where $P = [p_{10},\dots,p_{1n},p_{20},\dots,p_{2n},p_{M1},\dots,p_{Mn}]^T$ (16)

And $H(\gamma) = [\mathscr{b}_{1,0}(\gamma),\dots,\mathscr{b}_{1,n}(\gamma),\mathscr{b}_{2,0}(\gamma),\dots,\mathscr{b}_{2,n}(\gamma),$

$$\dots,\mathscr{b}_{M,1}(\gamma),\dots,\mathscr{b}_{M,n}(\gamma)]^T \tag{17}$$

We approximate φ as

$$\varphi(\gamma) = \sum_{m=1}^{M} \sum_{k=0}^{n} p_{m,k}\, \mathscr{b}_{m,k}(\gamma) = P^T H(\gamma). \tag{18}$$

where P and $H(\gamma)$ are defined by Eq. (16) and Eq. (17) respectively.

To solve the Eq. (1), put φ from Eq. (18) in Eq. (1), gives

$$\left(a(\gamma) \int_0^{\gamma} \sum_{m=1}^{M} \sum_{k=0}^{n} p_{m,k}\, \mathscr{b}_{m,k}(\gamma)\, k(\gamma,\delta) d\delta + b(\gamma) \int_{\gamma}^{1} \sum_{m=1}^{M} \sum_{k=0}^{n} p_{m,k}\, \mathscr{b}_{m,k}(\gamma)\, k(\delta,\gamma) d\delta\right) = \zeta(\gamma), \tag{19}$$

or,

$$\sum_{m=1}^{M}\sum_{k=0}^{n} p_{m,k}\left(a(\gamma)\int_0^{\gamma} \mathscr{b}_{m,k}(\gamma)k(\gamma,\delta)d\delta + b(\gamma)\int_{\gamma}^{1} \mathscr{b}_{m,k}(\gamma)k(\delta,\gamma)d\delta\right) = \zeta(\gamma), \tag{20}$$

Collocating Eq. (20) at $M(n+1)$ points $\gamma_l \in [0,1]$, $l = 1,2,\dots,M(n+1)$, the above equations reduces to the following linear system of equations,

$$\sum_{m=1}^{M}\sum_{k=0}^{n} p_{m,k}\left(a(\gamma_l)\int_0^{\gamma_l} \mathscr{b}_{m,k}(\gamma_l)k(\gamma_l,\delta)d\delta + b(\gamma_l)\int_{\gamma_l}^{1} \mathscr{b}_{m,k}(\gamma_l)k(\delta,\gamma_l)d\delta\right) = \zeta(\gamma_l). \tag{21}$$

After solving the above system of equations, we obtained the unknown values $\widetilde{p_{m,k}}$, which are the approximate values of $p_{m,k}$.

IV. CONVERGENCE ANALYSIS

This section includes the results on convergence of the schemes describe in last section and we will estimate an error bound for the both schemes. For this, Banach space $X = C[0,1]$ is considered associated with the norm defined as:

$$\| \varphi(\gamma) \| = \max_{0\le\gamma\le1}|\varphi(\gamma)|.$$

Also, we define,

$$\Psi\big(\omega(\gamma)\big) = \int_0^{\gamma} k(\gamma,\delta)\,\omega(\delta)d\delta, \tag{22}$$

and $\Psi_1\big(\omega(\gamma)\big) = \int_{\gamma}^{1} k(\delta,\gamma)\,\omega(\delta)d\delta,$ (23)

Lemma 1. [10] Let $X = C([0,1])$ and the kernel $\pi(\gamma,\delta)$ defined in Ψ and Ψ_1 is either continuous *i.e.*, $sup_{\gamma,\delta}\big(k(\gamma,\delta)\big) = Q$ (say) or square summable kernel $\pi(\gamma,\delta)$ i.e., $\int_0^1\int_0^1 |k(\gamma,\delta)|^2 d\gamma d\delta = Q^2$, where Q is a constant. Then Ψ and Ψ_1 are bounded on $[0,1]$, *i.e.*, there exist two constants such that

$$\| \Psi(\omega(x)) \| \le \mathrm{C}\| \omega(x) \| \text{ and } \| \Psi_1(\omega(x)) \| \le \mathrm{C}^1\| \omega(x) \|. \tag{24}$$

4.1 Convergence for S1:

Theorem 2. Consider that all the functions of Eq. (1) belongs to $C[0,1]$ and the kernel $k(\delta,\gamma)$ either belongs to $C[0,1]\times C[0,1]$ or $L^2[0,1)$. Let $\varphi(\gamma)$ and $\mathcal{B}_n\big(\varphi_n(\gamma)\big) = \sum_{k=0}^{n}\varphi_n(k/n)\,\mathcal{B}_{n,k}(\gamma)$ be the exact and approximated solution by scheme 1 of Eq. (1) respectively. Then, $|| \varphi(\gamma) - \mathcal{B}_n\big(\varphi_n(\gamma)\big) \|$

$$\le \left(1 + \frac{(\mathrm{C}\|a(\gamma)\|+\|\mathrm{C}^1 b(\gamma)\|)}{(\mathrm{C}\|a(\gamma)\|-\mathrm{C}^1\|b(\gamma)\|)}\right)\frac{1}{8n}\| \varphi'' \|. \tag{25}$$

Proof: Suppose that $\mathcal{B}_n\big(\varphi(\gamma)\big) = \sum_{k=0}^{n}\varphi(k/n)\,\mathcal{B}_{n,k}(\gamma)$ be the best approximation of $\varphi(\gamma)$. Then we have

$$\| \varphi(\gamma) - \mathcal{B}_n\big(\varphi_n(\gamma)\big) \| \le \| \varphi(\gamma) - \mathcal{B}_n\big(\varphi(\gamma)\big) \| + \| \mathcal{B}_n(\varphi(\gamma)) - \mathcal{B}_n\big(\varphi_n(\gamma)\big) \| \tag{26}$$

From relation given by Eq. (8), we obtain

$$\| \varphi(\gamma) - \mathcal{B}_n\big(\varphi(\gamma)\big) \| \le \frac{1}{8n}\| \varphi'' \|. \tag{27}$$

Therefore,

$$\| \varphi(\gamma) - \mathcal{B}_n\big(\varphi_n(\gamma)\big) \| \le \frac{1}{8n}\| \varphi'' \| + \| \mathcal{B}_n\big(\varphi(\gamma)\big) - \mathcal{B}_n\big(\varphi_n(\gamma)\big) \|. \tag{28}$$

Now, it is enough to estimate a bound for $\| \mathcal{B}_n(\varphi(\gamma)) - \mathcal{B}_n\big(\varphi_n(\gamma)\big) \|$. Theorem 1 implies that for any $\varphi \in C[0,1]$ and for any $\varepsilon > 0$ there exists n such that, $\| B_n(\varphi) - \varphi \| < \varepsilon$.

So, we can write Eq. (1) as

$$a(\gamma)\int_0^{\gamma} \mathcal{B}_n\big(\varphi(\delta)\big)k(\gamma,\delta)d\delta + b(\gamma)\int_{\gamma}^{1} \mathcal{B}_n\big(\varphi(\delta)\big)k(\delta,\gamma)d\delta = \zeta(\gamma), \tag{29}$$

If we substitute $\mathcal{B}_n(\varphi_n(\delta))$ instead of $\mathcal{B}_n(\varphi(\delta))$ in Eq. (29) then RHS is replaced by a new function $\zeta_1(\gamma)$ (say). So, we have,

$$a(\gamma)\int_0^\gamma \mathcal{B}_n(\varphi_n(\delta))k(x,\delta)d\delta + b(\gamma)\int_\gamma^1 \mathcal{B}_n(\varphi_n(\delta))k(\delta,\gamma)d\delta = \zeta_1(\gamma). \quad (30)$$

Consequently, we have

$$a(\gamma)\int_0^\gamma \{\mathcal{B}_n(\varphi(\delta)) - \mathcal{B}_n(\varphi_n(\delta))\}k(\gamma,\delta)d\delta + b(\gamma)\int_\gamma^1 \{\mathcal{B}_n(\varphi(\delta)) - \mathcal{B}_n(\varphi_n(\delta))\}k(\delta,\gamma)d\delta$$

$$= \zeta(\gamma) - \zeta_1(\gamma). \quad (31)$$

or,

$$\| a(\gamma)\int_0^\gamma \{\mathcal{B}_n(\varphi(\delta)) - \mathcal{B}_n(\varphi_n(\delta))\}k(\gamma,\delta)d\delta + b(\gamma)\int_\gamma^1 \{\mathcal{B}_n(\varphi(\delta)) - \mathcal{B}_n(\varphi_n(\delta))\}k(\delta,\gamma)d\delta \|$$

$$\leq \| \zeta(\gamma) - \zeta_1(\gamma) \|. \quad (32)$$

Or we can write,

$$\| a(\gamma)\Psi\{\mathcal{B}_n(\varphi(\gamma)) - \mathcal{B}_n(\varphi_n(\gamma))\} + b(\gamma)\Psi_1\{\mathcal{B}_n(\varphi(\gamma)) - \mathcal{B}_n(\varphi_n(\gamma))\} \| \leq \| \zeta(\gamma) - \zeta_1(\gamma) \|.$$

Using lemma 1, we obtain

$$\| \mathcal{B}_n(\varphi(\gamma)) - \mathcal{B}_n(\varphi_n(\gamma)) \| (\mathrm{C}\| a(\gamma) \| - \mathrm{C}^1\| b(\gamma) \|) \leq \| \zeta(\gamma) - \zeta_1(\gamma) \|.$$

$$\| \mathcal{B}_n(\varphi(\gamma)) - \mathcal{B}_n(\varphi_n(\gamma)) \|$$

$$\leq \frac{1}{(\mathrm{C}\|a(\gamma)\| - \mathrm{C}^1\|b(\gamma)\|)} \| \zeta(\gamma) - \zeta_1(\gamma) \|. \quad (33)$$

For finding a bound for,$\| \zeta(\gamma) - \zeta_1(\gamma) \|$, we assume

$$a(\gamma)\int_0^\gamma \varphi(\delta)k(\gamma,\delta)d\delta + b(\gamma)\int_\gamma^1 \varphi(\delta)k(\gamma,\delta)d\delta = \zeta(\gamma),$$

and, $a(\gamma)\int_0^\gamma \mathcal{B}_n(\varphi(\delta))k(\gamma,\delta)d\delta + \quad b(\gamma)\int_\gamma^1 \mathcal{B}_n(\varphi(\delta))k(\gamma,\delta)d\delta = \zeta_1(\gamma).$

So that,

$$\zeta(\gamma) - \zeta_1(\gamma) = a(\gamma)\int_0^\gamma (\mathcal{B}_n(\varphi(\delta)) - \varphi(\delta))k(\gamma,\delta)d\delta + b(\gamma)\int_\gamma^1 (\mathcal{B}_n(\varphi(\delta)) - \varphi(\delta))k(\gamma,\delta)d\delta,$$

$$\| \zeta(\gamma) - \zeta_1(\gamma) \| = \| a(\gamma)\int_0^\gamma (\mathcal{B}_n(\varphi(\delta)) - \varphi(\delta))k(\gamma,\delta)d\delta + b(\gamma)\int_\gamma^1 (\mathcal{B}_n(\varphi(\delta)) - \varphi(\delta))k(\gamma,\delta)d\delta \|,$$

Using lemma 1,

$$\| \zeta(\gamma) - \zeta_1(\gamma) \| \leq (\mathrm{C}\| a(\gamma) \| + \| \mathrm{C}^1 b(\gamma) \|) \| \Psi_1 \| \| \mathcal{B}_n(\varphi(\delta)) - \varphi(\delta) \|. \quad (34)$$

Now using Eq. (8), we obtain

$$\| \zeta(\gamma) - \zeta_1(\gamma) \| \leq (\mathrm{C}\| a(\gamma) \| + \| \mathrm{C}^1 b(\gamma) \|)\frac{1}{8n} \| \varphi'' \|, \quad (35)$$

By Eq. (28), (33), and (35), we conclude the required result.

4.2 Convergence for S2:

Theorem 3. Consider that all the functions of Eq. (1) belongs to $C[0,1]$ and the kernel $k(\delta,\gamma)$ either belongs to $C[0,1]\times C[0,1]$ or $L^2[0,1)$. Let $\varphi(\gamma)$ and $\varphi_{M,n}(\gamma) = \sum_{m=1}^{M}\sum_{k=0}^{n} p_{m,k}\mathcal{b}_{m,k}(\gamma) = P^T H(\gamma)$ be the exact and approximated solution by scheme 2 of Eq. (1) respectively. Suppose$\mathrm{C}\| a(\gamma) \| < \mathrm{C}^1\| b(\gamma) \|$. Then,

$$\lim_{n\to\infty} \| \varphi(\gamma) - \varphi_{M,n}(\gamma) \| = 0.$$

Proof: Substituting $\varphi_{M,n}(\gamma)$ in Eq. (1), results in

$$a(\gamma)\int_0^\gamma \varphi_{M,n}(\delta)k(\gamma,\delta)d\delta + b(\gamma)\int_\gamma^1 \varphi_{M,n}(\delta)k(\delta,\gamma)d\delta = \zeta(\gamma), \quad (36)$$

Now we have,

$$a(\gamma)\int_0^\gamma (\varphi(\delta) - \varphi_{M,n}(\delta))k(\gamma,\delta)d\delta + b(\gamma)\int_\gamma^1 (\varphi(\delta) - \varphi_{M,n}(\delta))k(\delta,\gamma)d\delta = 0, \quad (37)$$

$$\| a(\gamma)\int_0^\gamma \left(\varphi(\delta) - \varphi_{M,n}(\delta)\right)k(\gamma,\delta)d\delta \| - \| b(\gamma)\int_\gamma^1 (\varphi(\delta) - \varphi_{M,n}(\delta))k(\delta,\gamma)d\delta \| \leq 0, \quad (38)$$

Using lemma 1, above equation reduces to,

$$\mathrm{C}\| a(\gamma) \| \| \varphi(\gamma) - \varphi_{M,n}(\gamma) \| - \mathrm{C}^1\| b(\gamma) \| \| \varphi(\gamma) - \varphi_{M,n}(\gamma) \| \leq 0,$$

$$(\mathrm{C}\| a(\gamma) \| - \mathrm{C}^1\| b(\gamma) \|) \| \varphi(\gamma) - \varphi_{M,n}(\gamma) \| \leq 0,$$

taking $\tau = \frac{C^1\|b(\gamma)\|}{C\|a(\gamma)\|}$, we have

$(1-\tau) \parallel \varphi(\gamma) - \varphi_{M,n}(\gamma) \parallel \leq 0,$

since $0 < \tau < 1$, therefore

$\lim_{n\to\infty} \parallel \varphi(\gamma) - \varphi_{M,n}(\gamma) \parallel = 0.$ (39)

V. RESULTS, DISCUSSION AND NUMERICAL STABILITY:

In this section, a numerical practice of the proposed method will be done and its veracity will be verified. Also, the stability will be shown by implementing it on the GAIEs with known solutions.

The accuracy of both proposed methods is demonstrated by the maximum absolute error defined by

$$E = |\varphi(\gamma) - \varphi_n(\gamma)|.$$

Here, $\varphi_n(\gamma)$ is the approximated solution and $\varphi(\gamma)$ is the exact solution of GAIEs.

In the following considered examples, if we rewrite the Eq. (1) in operator form as

$$\chi\big(\varphi(\gamma)\big) = \zeta(\gamma).$$ (40)

Here, $\varphi(\gamma)$ denotes the exact solution of Eq. (40), ${\varphi_n}^{\delta}(\gamma)$ is the noisy approximated function corresponding to the ε times random error to $\varphi(\gamma)$ such that ${\varphi_n}^{\delta}(\gamma)$ is the approximate solution of the problem,

$$\chi\big(\varphi(\gamma)\big) = \zeta(\gamma) + \varepsilon\delta.$$ (41)

${\varphi_n}^{0}(\gamma)$ denotes the approximate solution of Eq. (41) without noise. ε is a sufficiently small constant and δ is a uniform random variable within $[-1,1]$. By applying both methods mentioned above for noisy analysis, the solution ${\varphi_n}^{\delta}(\gamma)$ of Eq. (41) is obtained. The test example is solved and analyzed. The obtained results are demonstrated in the different figures which represent the comparison between:

i. Exact solution $\varphi(\gamma)$, the approximate solution without noise ${\varphi_n}^{0}(\gamma)$, the approximate solution with noise ${\varphi_n}^{\delta 1}(\gamma)$ and ${\varphi_n}^{\delta 2}(\gamma)$ with random noise $\delta 1$ and $\delta 2$ respectively.

ii. An approximate solution with random noise ${\varphi_n}^{\delta 1}(\gamma)$ and ${\varphi_n}^{\delta 2}(\gamma)$.

Test example 1 This example has been taken from [10], in which exact solution is in fraction power of γ. For this we have, $a(\gamma) = b(\gamma) = 1$, $k(\gamma,\delta) = \frac{1}{(\gamma-\delta)^{1/2}}$and

$$\zeta(\gamma) = \frac{\sqrt{\pi}\gamma^{\frac{5}{6}}\Gamma(-\frac{5}{6})}{\Gamma(-\frac{1}{3})} + \frac{\sqrt{\pi}\gamma^{\frac{5}{6}}\Gamma(\frac{4}{3})}{\Gamma(\frac{11}{6})} + \frac{6}{5_2}F_1[-\frac{5}{6},\frac{1}{2},\frac{1}{6},\gamma].$$

For this case, the exact solution is given by $\varphi(\gamma) = \gamma^{1/3}$. This problem is solved by the scheme S1 described in section 3 with $n = 1,2,3$ and by scheme S2 with $n = 1,2,3$ and $M = 5,10,15,20$. The obtained maximum absolute errors are shown in Table 1. According to the Table 1, we achieved maximum absolute error in order $o(10^{-5})$ which shows the superiority of scheme S2 over S1 and the method presented in [10] (Table-2). The obtained maximum absolute errors in [10] is in $o(10^{-2})$ with$n = 3,5$ (Table-1).

As the exact solution in this case is not smooth so scheme S1 doesn't provide the better accuracy and fast convergence (Figure 1). This is also clear from Eq. (8) which states that the Bernstein approximation converges to the exact solution when the solution is twice continuously differentiable i.e.,$\varphi \in C^2[0,1]$. Also, the available methods for GAIEs are failed to achieve an approximation close to the exact solution when the exact solution is not smooth [10]. This is the beauty of taking the basis functions as hybrid Bernstein polynomial and the approximation converges to exact solution rapidly even when the exact solution is not smooth (Figure 1).

Table 1. Maximum absolute error by $S1$ and $S2$ for test example 2.

$nS1$	$S2$	
	M = 5	**M = 10 M = 15 M = 20**

.932	0.119	$)8E-3$	$2.26E-3$	$9.05E-4$	$5.44E-4$
$37.49E-2$		$6.49E-4$	$3.54E-4$	$1.54E-4$	$7.81E-5$
		$?9E-4$	$1.43E-4$	$6.64E-5$	$3.88E-5$

Table 2. Maximum absolute error for test example 2 using different polynomials [10].

	ıcobi	***ɐgendre***	***hebyshev***	***egenbauer***
	$\mathbf{37.499E-2}$	$\mathbf{7.499E-2}$	$\mathbf{7.499E-2}$	$\mathbf{7.499E-2}$
5	$5.781\boldsymbol{E-2}$	$5.781\boldsymbol{E-2}$	$5.781\boldsymbol{E-2}$	$\mathbf{5.781E-2}$

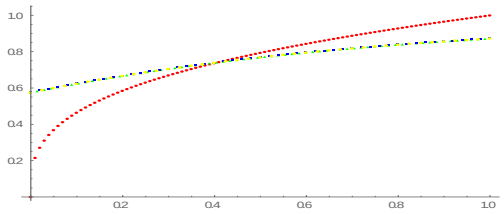

Fig 1. Exact solution $\varphi(\gamma)$(red circle), the approximate solution without noise ${\varphi_n}^0(\gamma)$ (green triangle), approximate solution with noises ${\varphi_n}^{\delta 1}(\gamma)$(blue square) and ${\varphi_n}^{\delta 2}(\gamma)$ (yellow circle) for test example 1 by Scheme S1.

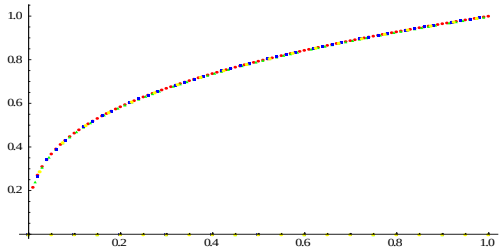

Fig 2. Exact solution $\varphi(\gamma)$(red circle), the approximate solution without noise ${\varphi_n}^0(\gamma)$ (green triangle), approximate solution with noises ${\varphi_n}^{\delta 1}(\gamma)$ (blue square) and ${\varphi_n}^{\delta 2}(\gamma)$ (yellow circle) for test example 1 by Scheme S2.

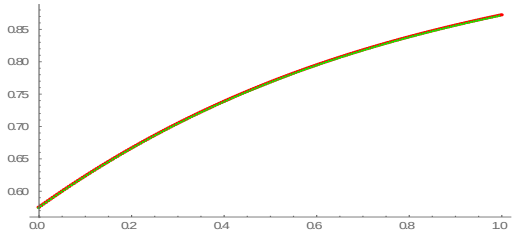

Fig. 3The approximate solutions with noises ${\varphi_n}^{\delta 1}(\gamma)$ (red circle) and ${\varphi_n}^{\delta 2}(\gamma)$ (green triangle) for test example 1 by Scheme S1.

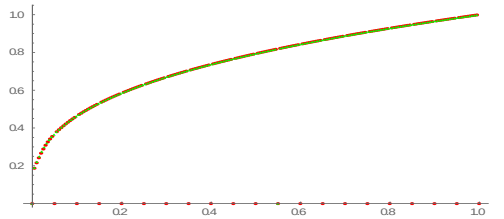

Fig. 4 The approximate solutions with noises ${\varphi_n}^{\delta 1}(\gamma)$ (red circle) and ${\varphi_n}^{\delta 2}(\gamma)$ (green triangle) for test example 1 by Scheme S2.

VI. APPLICATION IN TOMOGRAPHY:

The interior of an object was imaged by Abel [1] in terms of inversion in 1826, which resulted in Abel inversion techniques. Abel analytically solved the inversion of a cylindrically symmetric object. Before the invention of x-rays, there was not any develop the technique which images the interior of an object. Although the Radon transform is generally used in x-ray tomography. The Radon transform reduces to the Abel transform, which exactly reconstructs a cylindrically symmetrical object from a single x-ray radiograph. However, difficulties in utilizing the Abel transform

arise for several reasons that include the physics of x-ray radiography, as well as properties of the Abel transform itself, as it applies to this problem.

Abel transform describes the relation between the result distribution of the emission coefficient $\varepsilon(r)$ and measured intensity $I(y)$. Reconstruction of the emission coefficient from its projection is known as Abel inversion.

The relation between the emission coefficients $\varepsilon_\lambda(r)$ and intensity $I_\lambda(y)$ can be described as [14]

$$I_\lambda(y) = \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \varepsilon_\lambda(r)dr, \tag{42}$$

for a specific wavelength λ, here y denotes the displacement of the intensity profile from the line of plasma center, the radial distance from the center of the source $x^2 + y^2 = r^2$is r, and a is the source radius. It is assumed that $\varepsilon_\lambda(r)$ vanishes for $r > a$, and hence $I_\lambda(y)$ vanishes for the$mod(y) > a$. For simplicity, we take $a = 1.0$ in Eq. (42). By changing the variable of integration to r in Eq. (42), we obtain,

$$I(y) = 2\int_y^1 \frac{\varepsilon(r).r}{\sqrt{r^2-y^2}}dr. \tag{43}$$

By change of variables, Eq. (43) reduces to a special case of Eq. (1) when$a(\gamma) = 0,\ b(\gamma) = 1$,

$$\zeta(\gamma) = \int_\gamma^1 \frac{\phi(y)}{\sqrt{\gamma-y}}dy, \tag{44}$$

where $\zeta(\gamma) = I(\sqrt{\gamma})$ and $\phi(y) = (\varepsilon\sqrt{\gamma})$.

Test example 2. Consider Eq. (44) with [14]

$\phi(y) = \frac{1}{2}(1 + 10y - 23y^2 + 12y^3)$, $0 \le y \le 1$,

$\zeta(\gamma) = \frac{8}{105}(1-\gamma)^{5/2}(19 + 72\gamma)$, $0 \le \gamma \le 1$.

We solve this problem with both schemes and obtain numerical results. In this case, the obtained maximum absolute error is $E = 2.868 \times 10^{-16}$ by S1 and $E = 5.270 \times 10^{-15}$ by S2. Also, we solve test example 2, with some random noise. Figures 5 and 6 illustrate that approximate solutions with and without noise almost coincide. For this test example, we plot the output results (Figs. 5-8) and also the maximum absolute errors (Figs. 9-10) which are defined as,

i.) Maximum absolute error E between the exact and approximate solution without noise,

$$E = |\varphi(\gamma) - \varphi_n(\gamma)|.$$

ii.) Maximum absolute error $E1$ between the exact and approximate solution with noise $\delta 1$,

$$E1 = |\varphi(\gamma) - \varphi_n{}^{\delta 1}(\gamma)|.$$

iii.) Maximum absolute error $E2$ between the exact and approximate solution with noise $\delta 2$,

$$E2 = |\varphi(\gamma) - \varphi_n{}^{\delta 2}(\gamma)|.$$

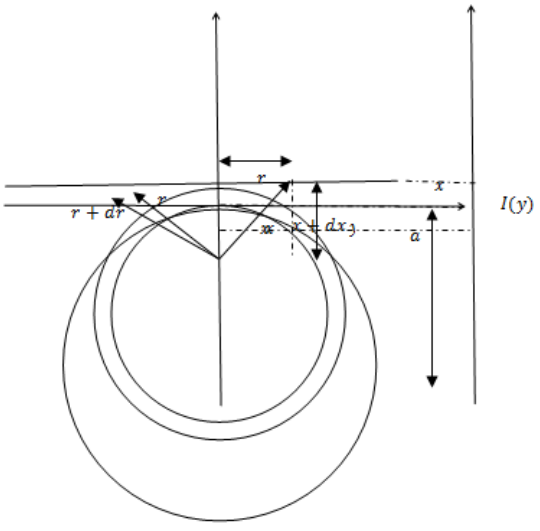

Fig. A. Geometrical interpretation of the Abel transform in two dimensions with radius *a*[14].

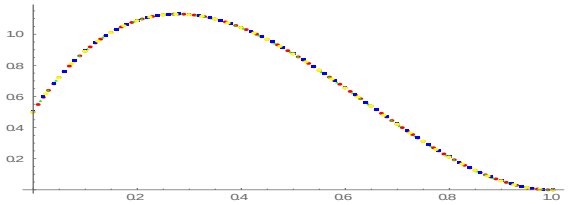

Fig 5: Exact solution $\varphi(\gamma)$(red circle), the approximate solution without noise ${\varphi_n}^0(\gamma)$ (green triangle), approximate solution with noises ${\varphi_n}^{\delta 1}(\gamma)$ (blue square) and ${\varphi_n}^{\delta 2}(\gamma)$ (yellow circle) for test example 2 by Scheme S1.

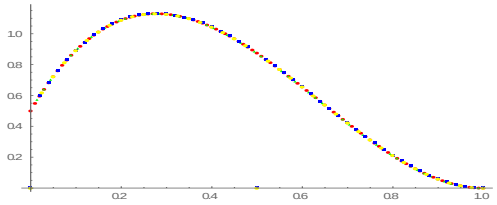

Fig 6: Exact solution $\varphi(\gamma)$(red circle), the approximate solution without noise ${\varphi_n}^0(\gamma)$ (green triangle), approximate solution with noises ${\varphi_n}^{\delta 1}(\gamma)$ (blue square) and ${\varphi_n}^{\delta 2}(\gamma)$ (yellow circle) for test example 2 by Scheme S2.

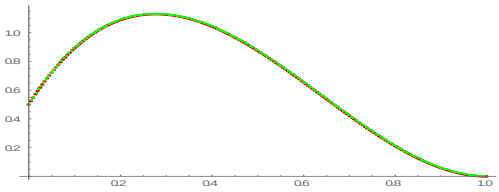

Fig. 7: The approximate solutions with noises ${\varphi_n}^{\delta 1}(\gamma)$ (red circle) and ${\varphi_n}^{\delta 2}(\gamma)$ (green triangle) for test example 2 by Scheme S1.

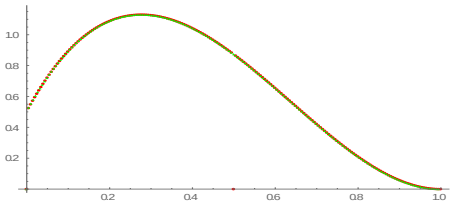

Fig. 8: The approximate solutions with noises ${\varphi_n}^{\delta 1}(\gamma)$ (red circle) and ${\varphi_n}^{\delta 2}(\gamma)$ (green triangle) for test example 2 by Scheme S2.

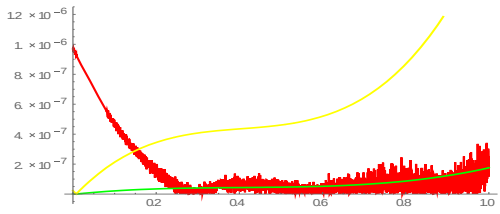

Fig. 9: Obtained maximum absolute errors $10^8 E(red)$, $E1(green)$, $10^3 E2(yellow)$ for test example 2 by S1.

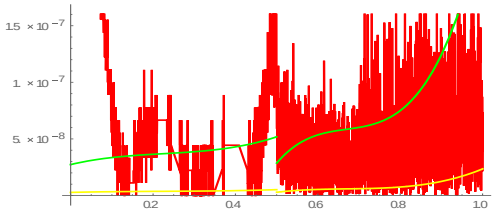

Fig. 10: Obtained maximum absolute errors $10^8 E(red)$, $E1(green)$, $10^1 E2(yellow)$for test example 2 by S2.

VII. CONCLUSION:

A new and simple approach based on Bernstein polynomials (scheme 1) is presented coupled with collocation method to approximate the numerical solution of GAIEs. In order to obtain the better accuracy, when the exact solution is not smooth or twice continuously differentiable, hybrid Bernstein Block-Pulse function is used as basis functions (in scheme 2) and better results have been obtained. The simplicity of this method is that it converts the GAIEs into algebraic equations which can be rapidly solved by computation. The theoretical results describing the convergence of the schemes S1 and S2 are also established. Numerical experiments are added to demonstrate the accuracy of lower order approximations. It has been noticed that both schemes work well and provide better approximation. Moreover, scheme 2 provides better results in comparison of scheme 1 and has very high accuracy. It proves its superiority over the S1 and over available methods in the literature [7, 10].

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