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SUFFICIENT CONDITION FOR A NIL-CLEAN ELEMENT TO BE CLEAN IN A CERTAIN SUBRING OF$M_3(\mathbb{Z})$

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Abstract: Diesl has proved that a nil-clean ring is clean. However, not all nil-clean element of any ring is clean as showed by Andrica by providing counter examples in 2 x 2 matrices over $\mathbb{Z}$. The objective of this study is to determine sufficient condition for a nil-clean element tobe clean in a certain subring of $\boldsymbol{M_3}(\mathbb{Z})$. The two main methods are constructing certain subring, namely $\boldsymbol{X_3}(\mathbb{Z})$, of $\boldsymbol{M_3}(\mathbb{Z})$ and then identifying idempotent and nilpotent elements in $\boldsymbol{X_3}(\mathbb{Z})$. This construction provides examples as the extension of those matrices founded by Andrica in the sense of matrix order and the different form of those matrices. The methods are used in finding the sufficient condition for nil-clean elements to be clean in a certain subring of $\boldsymbol{M_3}(\mathbb{Z})$. By this finding, we follow up the previous researches especially from Diesl and Andrica. As the application, it is provided nil-clean elements in $\boldsymbol{X_3}(\mathbb{Z})$ which are clean and some other elements which are not clean.

Keywords: nil-clean ring, clean ring,nil-clean element, clean element, matrices

I. INTRODUCTION

Let R be a ring with unity. An element a is called a unit if there exist $a^{-1} \in R$ such that $aa^{-1} = a^{-1}a = 1$.If $a^2 = a$, then a is called idempotent. If there exist $n \in \mathbb{N}$such that $a^n = 0$, then a is called nilpotent.If$a = 1 + b$for some nilpotent$b$, then a is called unipotent. Let$U(R)$denotes the set of all units in$R$, $Id(R)$denotes the set of all idempotent elementsin$R$, and$Nil(R)$denotes the set of all nilpotent elements in$R$. If$a \in R$ can be written as$a = e + u$for some$e \in Id(R)$ and $u \in U(R)$, then $a$is said to be clean. Clean rings were first introduced in[1] as an extension of exchange rings. We say that$a \in R$ is nil-clean if $a = e + n$for some$e \in Id(R)$ and $n \in Nil(R)$[2]. A ring R is said to be clean (or nil-clean) if all of its elements are clean (or nil-clean, respectively). Any field$F$ is a clean ring since $0 = -1 + 1$ and $0 \neq a = a + 0$, where $-1, a \in U(F), 0,1 \in Id(F)$. Field $\mathbb{F}_2$ is nil-clean ring since $0 = 0 + 0, 1 = 1 + 0$, where $0,1 \in Id(\mathbb{F}_2), 0 \in Nil(\mathbb{F}_2)$. Dubrovin valuation ring $Q(\mathbb{Z})$ of simple artinian ring $Q(\mathbb{R})$ is not nil-clean since $2 \in Q(\mathbb{Z})$, $2 \neq e + n, \forall e \in Id(Q(\mathbb{Z})), \forall n \in Nil(Q(\mathbb{Z}))$.

Every nil-clean ring is clean [3]. However, not every nil-clean element of any ring is clean. [2] give an example of a nil-clean element which is not clean that is the matrix $\begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -6 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}$ in $M_2(\mathbb{Z})$.

In this paper,we have extended the result of[2] to 3×3 matrix over integers, in a certain subring of $M_3(\mathbb{Z})$. We also find the sufficient condition for a nil-clean element to be clean in the certain subring of $M_3(\mathbb{Z})$.

II. PRELIMINARIES

Many studies about the connections between nil-clean elements and clean elements are confined to 2×2 matrices. To extend the study more about the connections between nil-clean elements and clean elements we need to know the previous results from the literature. For convenience to the reader, we state in the following some results from the literature related to the results presented in this paper:

Theorem 2.1 – Pell's equation [4] Let D be an integer which is not a perfect square. Then the following equation:

$$u^2 - Dv^2 = 1$$

has infinitely many nonnegative integer solutions, and the general solutions$(u_n, v_n)_{n\geq 0}$ are given by:

$$u_{n+1} = u_1 u_n + D v_1 v_n,$$
$$v_{n+1} = v_1 u_n + u_1 v_n,$$

where (u_1, v_1) is the fundamental solution, i.e., the solution with $v_1 > 0$ minimal.

Lemma 2.2 [2]The nontrivial idempotent elements in $M_2(\mathbb{Z})$ are matrices of the form $\begin{bmatrix} \alpha+1 & u \\ v & -\alpha \end{bmatrix}$ where $\alpha^2 + \alpha + uv = 0$.

Lemma 2.3[2]The nilpotent elements in $M_2(\mathbb{Z})$ are matrices of the form $\begin{bmatrix} \beta & x \\ y & -\beta \end{bmatrix}$ where $\beta^2 + \beta + xy = 0$.

Preposition 2.4 [2] Nil-clean upper triangular matrices which is neither unipotent nor nilpotent element, is idempotent element, so the matrices is clean.

Theorem 2.5 [2]Let $C = \begin{bmatrix} r+1 & u+x \\ v+y & -r \end{bmatrix}$ be a nontrivial nil-clean matrix and let $E = \begin{bmatrix} \gamma+1 & s \\ t & -\gamma \end{bmatrix}$ be nontrivial idempotent matrix. Then, $C - E$ is unit in $M_2(\mathbb{Z})$ with $det(C - E) = 1$ if and only if

$$X^2 - (1+4\delta)Y^2 = 4(v+y)^2(2r+1)^2(\delta^2 + 2\delta + 2)$$

with

$$X = (2r+1)[-(1+4\delta)t + (2\delta+3)(v+y)]$$

and

$Y = 2(v+y)^2 s + (2r^2 + 2r + 1 + 2\delta)t - (2\delta+3)(v+y)$.

Further, $C - E$ is unit in $M_2(\mathbb{Z})$ with $\det(C - E) = -1$ if and only if

$$X^2 - (1+4\delta)Y^2 = 4(v+y)^2(2r+1)^2\delta(\delta - 2)$$

with

$$X = (2r+1)[-(1+4\delta)t + (2\delta-1)(v+y)]$$

and

$Y = 2(v+y)^2 s + (2r^2 + 2r + 1 + 2\delta)t - (2\delta-1)(v+y)$.

Proposition 2.6[3] Let R be a ring and let I be any nil ideal of R. Then R is nil-clean if and only if R/I is nil-clean.

Lemma 2.7[5]Every complete matrix ring $M_n(R)$ over a clean ring is clean.

Theorem 2.8 [3]Every nil-clean ring is clean.

III. RESULT AND DISCUSSION

Idempotent and nilpotent elements in a certain subringof $M_3(\mathbb{Z})$

Consider a matrix $M \in M_n(\mathbb{Z})$. Let $p_M(\lambda)$ be the characteristic polynomial of M. Then the coefficient of λ^{n-1} of $p_M(\lambda)$ is $-tr(M)$ whereas the constant term of $p_M(\lambda)$ is $(-1)^n \det(M)$, that is

$$p_M(\lambda) = \sum_{k=0}^{n} S_k(M)\, \lambda^{n-k}$$

where $S_k(M)$ is the k-$minor$ of M[6].

Now, we define and show that

$$X_3(\mathbb{Z}) := \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \mid a_{11}, a_{13}, a_{22}, a_{31}, a_{33} \in \mathbb{Z} \right\}$$

is a subring of $M_3(\mathbb{Z})$. The process is described in the following subring test.

$X_3(\mathbb{Z}) \neq \emptyset$, since$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in X_3(\mathbb{Z})$.

Now, let $A = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}$ and

$B = \begin{bmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & 0 \\ b_{31} & 0 & b_{33} \end{bmatrix}$be any elements in $X_3(\mathbb{Z})$.

We have

$$A - B = \begin{bmatrix} a_{11}-b_{11} & 0 & a_{13}-b_{13} \\ 0 & a_{22}-b_{22} & 0 \\ a_{31}-b_{31} & 0 & a_{33}-b_{33} \end{bmatrix}$$

and

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{13}b_{31} & 0 & a_{11}b_{13} + a_{13}b_{33} \\ 0 & a_{22}b_{22} & 0 \\ a_{31}b_{11} + a_{33}b_{31} & 0 & a_{31}b_{13} + a_{33}b_{33} \end{bmatrix}$$ are also in $X_3(\mathbb{Z})$.

Then, by the subring test, we obtain that$X_3(\mathbb{Z})$ is a subring of $M_3(\mathbb{Z})$.

Moreover, For any matrix

$$A = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \in X_3(\mathbb{Z}),$$

the characteristic polynomial of A is

$$p_A(t) = t^3 - tr(A)t^2 + S_2(A)t - \det(A)$$

where

- $\det A \;= a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31}$,
- $tr(A) \;= a_{11} + a_{22} + a_{33}$,
- $S_2(A) \;= \begin{vmatrix} a_{22} & 0 \\ 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix}$

$= a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31} + a_{11}a_{22}$.

By the Cayley-Hamilton theorem, we obtain

$$0 = p_A(A) = A^3 - tr(A)A^2 + S_2(A)A - \det(A)\, I_3$$

(see for example [7]).

To determine the nil-clean elements in $X_3(\mathbb{Z})$, we firstlydetermine the general forms of the idempotent and nilpotent elementsin $X_3(\mathbb{Z})$.

Lemma 3.1 The nontrivial idempotent elements in$X_3(\mathbb{Z})$are matricesof the form $\begin{bmatrix} 1-i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$ or $\begin{bmatrix} \alpha+1 & 0 & u \\ 0 & i & 0 \\ v & 0 & -\alpha \end{bmatrix}$with$i \in \{0,1\}$ and$\alpha, u, v \in \mathbb{Z}$ satisfy $\alpha^2 + \alpha + uv = 0$.

Proof:

Let $i \in \{0,1\}$and $\alpha, u, v \in \mathbb{Z}$ such that$\alpha^2 + \alpha + uv = 0$. Then $\begin{bmatrix} 1-i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$ and $\begin{bmatrix} \alpha+1 & 0 & u \\ 0 & i & 0 \\ v & 0 & -\alpha \end{bmatrix}$ are idempotent elements in $X_3(\mathbb{Z})$ because

$$\begin{bmatrix} 1-i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1-i \end{bmatrix}^2 = \begin{bmatrix} 1-i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$$

and

$$\begin{bmatrix}\alpha+1 & 0 & u\\ 0 & i & 0\\ v & 0 & -\alpha\end{bmatrix}^2=\begin{bmatrix}\alpha+1 & 0 & u\\ 0 & i & 0\\ v & 0 & -\alpha\end{bmatrix}.$$

Conversely, let $E=\begin{bmatrix}a_{11} & 0 & a_{13}\\ 0 & a_{22} & 0\\ a_{31} & 0 & a_{33}\end{bmatrix}\in X_3(\mathbb{Z})$ be an idempotent. Then $E^2=E$, that is,

$$\begin{bmatrix}a_{11} & 0 & a_{13}\\ 0 & a_{22} & 0\\ a_{31} & 0 & a_{33}\end{bmatrix}\begin{bmatrix}a_{11} & 0 & a_{13}\\ 0 & a_{22} & 0\\ a_{31} & 0 & a_{33}\end{bmatrix}=\begin{bmatrix}a_{11} & 0 & a_{13}\\ 0 & a_{22} & 0\\ a_{31} & 0 & a_{33}\end{bmatrix}.$$

Expanding the product on the left side, we obtain

$$\begin{bmatrix}a_{11}^2+a_{13}a_{31} & 0 & (a_{11}+a_{33})a_{13}\\ 0 & a_{22}^2 & 0\\ (a_{11}+a_{33})a_{31} & 0 & a_{31}a_{13}+a_{33}^2\end{bmatrix}=\begin{bmatrix}a_{11} & 0 & a_{13}\\ 0 & a_{22} & 0\\ a_{31} & 0 & a_{33}\end{bmatrix}.$$

By comparing the entries of the matrices, we have the following system of equations:

$$a_{11}^2+a_{13}a_{31}=a_{11} \quad (1)$$
$$(a_{11}+a_{33})a_{13}=a_{13} \quad (2)$$
$$a_{22}^2=a_{22} \quad (3)$$
$$(a_{11}+a_{33})a_{31}=a_{31} \quad (4)$$
$$a_{31}a_{13}+a_{33}^2=a_{33} \quad (5)$$

By equation (3) we obtain $a_{22}\in\{0,1\}$. By equations (2) and (4) we have either $a_{11}+a_{33}=1$, or $a_{11}+a_{33}\neq 1$ but $a_{13}=0$ and $a_{31}=0$.

Case 1: For $a_{11}+a_{33}=1$.
By squaring both sides of the equation, we obtain

$$\boldsymbol{a_{11}^2+a_{33}^2+2a_{11}a_{33}=1} \quad (\boldsymbol{6})$$

By taking the sum of equations (1) and (5), we obtain

$$\boldsymbol{a_{11}^2+a_{33}^2+2a_{13}a_{31}=a_{11}+a_{33}=1}. \quad (\boldsymbol{7})$$

Then by comparing (6) and (7), we obtain $a_{11}a_{33}=a_{13}a_{31}$.
Now by letting $a_{11}=\alpha+1$, we have

$$a_{33}=1-(\alpha+1)=-\alpha.$$

Let $a_{13}=u$ and $a_{31}=v$. It follows then that $E=\begin{bmatrix}\alpha+1 & 0 & u\\ 0 & i & 0\\ v & 0 & -\alpha\end{bmatrix}$ with $i\in\{0,1\}$, $\alpha,u,v\in\mathbb{Z}$ and $(\alpha+1)(-\alpha)=uv$, that is, $\alpha^2+\alpha+uv=0$.

Case 2: For $a_{11}+a_{33}\neq 1$.
In this case, $a_{13}=0$ and $a_{31}=0$. Therefore, $a_{11}^2=a_{11}$ and $a_{33}^2=a_{33}$, that is, $a_{11},a_{33}\in\{0,1\}$. It follows that the possible idempotent elements in $X_3(\mathbb{Z})$ are the trivial ones, namely, 0_3 and I_3, and the nontrivial ones, namely, $\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$.

This completes the proof of Lemma 3.1.

Lemma 3.2 The nilpotent elements in $X_3(\mathbb{Z})$ are matrices of the form $\begin{bmatrix}\beta & 0 & x\\ 0&0&0\\ y & 0 & -\beta\end{bmatrix}$ with $\beta,x,y\in\mathbb{Z}$ satisfying $\beta^2+xy=0$.

Proof: Note that $\begin{bmatrix}\beta & 0 & x\\ 0&0&0\\ y & 0 & -\beta\end{bmatrix}$, where $\beta,x,y\in\mathbb{Z}$ satisfy $\beta^2+xy=0$, is a nilpotent matrix because $\begin{bmatrix}\beta & 0 & x\\ 0&0&0\\ y & 0 & -\beta\end{bmatrix}^2=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$.

Conversely, let $N=\begin{bmatrix}a_{11} & 0 & a_{13}\\ 0 & a_{22} & 0\\ a_{31} & 0 & a_{33}\end{bmatrix}\in X_3(\mathbb{Z})$. If N is nilpotent, that is, there exist a positive integer k such that $N^k=0$, then $p_N(t)=t^3$. We obtain the following system of equations:

$$0=\det(N)=a_{11}a_{22}a_{33}-a_{13}a_{22}a_{31} \quad (8)$$
$$0=tr(N)=a_{11}+a_{22}+a_{33} \quad (9)$$
$$0=S_2(N)=a_{22}a_{33}+a_{11}a_{33}-a_{13}a_{31}+a_{11}a_{22} \quad (10)$$

In order to have N as nilpotent, we must have $N^3=0_3$. By expanding N^3 and comparing its entries with 0, we obtain in particular the equation $a_{22}^3=0$. Thus, $a_{22}=0$. By substituting $a_{22}=0$ into equations (9) and (10), we obtain $a_{11}+a_{33}=0$ and $a_{11}a_{33}=a_{13}a_{31}$, respectively. Let $a_{11}=\beta, a_{13}=x, a_{31}=y$. Then $N=\begin{bmatrix}\beta & 0 & x\\ 0&0&0\\ y & 0 & -\beta\end{bmatrix}$ with $\beta,x,y\in\mathbb{Z}$ such that $\beta^2+xy=0$. This completes the proof.

IV. NIL-CLEAN MATRICES IN $X_3(\mathbb{Z})$.

By Lemmas 3.1 and 3.2, we obtain the general form of the nil-clean elements in $X_3(\mathbb{Z})$ as follows:

$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\begin{bmatrix}\beta & 0 & x\\ 0&0&0\\ y & 0 & -\beta\end{bmatrix}=\begin{bmatrix}\beta & 0 & x\\ 0&1&0\\ y & 0 & -\beta\end{bmatrix},$$

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}+\begin{bmatrix}\beta & 0 & x\\ 0&0&0\\ y & 0 & -\beta\end{bmatrix}=\begin{bmatrix}\beta+1 & 0 & x\\ 0&0&0\\ y & 0 & 1-\beta\end{bmatrix}, \text{ or}$$

$$\begin{bmatrix}\alpha+1 & 0 & u\\ 0 & i & 0\\ v & 0 & -\alpha\end{bmatrix}+\begin{bmatrix}\beta & 0 & x\\ 0&0&0\\ y & 0 & -\beta\end{bmatrix}=\begin{bmatrix}\alpha+\beta+1 & 0 & u+x\\ 0 & i & 0\\ v+y & 0 & -\alpha-\beta\end{bmatrix}$$

with $i,j\in\{0,1\}$, $\alpha,\beta,u,v,x,y\in\mathbb{Z}$ and satisfy $\alpha^2+\alpha+uv=0=\beta^2+xy$.

Now, observe that $\begin{bmatrix}\beta & 0 & x\\ 0&1&0\\ y & 0 & -\beta\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}+\begin{bmatrix}\beta-1 & 0 & x\\ 0&1&0\\ y & 0 & -\beta-1\end{bmatrix}$ and $\begin{bmatrix}\beta+1 & 0 & x\\ 0&0&0\\ y & 0 & 1-\beta\end{bmatrix}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\begin{bmatrix}\beta+1 & 0 & x\\ 0&-1&0\\ y & 0 & 1-\beta\end{bmatrix}$ are clean in $X_3(\mathbb{Z})$.

So $\begin{bmatrix}\beta & 0 & x\\ 0&1&0\\ y & 0 & -\beta\end{bmatrix}$ and $\begin{bmatrix}\beta+1 & 0 & x\\ 0&0&0\\ y & 0 & 1-\beta\end{bmatrix}$ are nil-clean elements which are also clean in $X_3(\mathbb{Z})$.

To find another form of nil-clean elements whis are also clean in $X_3(\mathbb{Z})$ we need variables $\alpha,\beta,u,v,x,y,i\in\mathbb{Z}$ with

$i \in \{0,1\}$and$\alpha^2+\alpha+uv=0=\beta^2+xy$ so, for each$\gamma,s,t\in\mathbb{Z}$and$j\in\{0,1\}$satisfying

(i) $\det\left(\begin{bmatrix}\alpha+\beta+1 & 0 & u+x\\ 0 & i & 0\\ v+y & 0 & -\alpha-\beta\end{bmatrix}-\begin{bmatrix}1-j & 0 & 0\\ 0 & j & 0\\ 0 & 0 & 1-j\end{bmatrix}\right)$
$=\det\left(\begin{bmatrix}\alpha+\beta+j & 0 & u+x\\ 0 & i-j & 0\\ v+y & 0 & -\alpha-\beta-1+j\end{bmatrix}\right)$
$=(i-j)\big(-(\alpha+\beta+j)(\alpha+\beta+1-j)-(u+x)(v+y)\big)\in\{-1,1\}$, or

(ii) $\det\left(\begin{bmatrix}\alpha+\beta+1 & 0 & u+x\\ 0 & i & 0\\ v+y & 0 & -\alpha-\beta\end{bmatrix}-\begin{bmatrix}\gamma+1 & 0 & s\\ 0 & j & 0\\ t & 0 & -\gamma\end{bmatrix}\right)$
$=\det\left(\begin{bmatrix}\alpha+\beta-\gamma & 0 & u+x-s\\ 0 & i-j & 0\\ v+y-t & 0 & -\alpha-\beta+\gamma\end{bmatrix}\right)$
$=(i-j)\big(-(\alpha+\beta-\gamma)^2-(u+x-s)(v+y-t)\big)\in\{-1,1\}$.

If$i-j=0$then the determinant equals $0\neq\pm1$.
If$i-j\neq0$then$i-j=\pm1$. So we have

(i) $\big(-(\alpha+\beta)(\alpha+\beta+1)-(u+x)(v+y)\big)\in\{-1,1\}$, or
(ii) $\big(-(\alpha+\beta-\gamma)^2-(u+x-s)(v+y-t)\big)\in\{-1,1\}$.

This is the sufficient condition for nil-clean elements to be clean in $X_3(\mathbb{Z})$.This conditions is in line with theorem 4 of[2].

Now we state the thoerem about sufficient conditions for nil-clean elements to be clean in $X_3(\mathbb{Z})$. For simplification, let $r=\alpha+\beta$ and $\delta=r^2+r+(v+y)(u+x)$.

Theorem 4.1

Let$C=\begin{bmatrix}r+1 & 0 & u+x\\ 0 & i & 0\\ v+y & 0 & -r\end{bmatrix}$be nontrivial nil-clean matrixandlet$E$be nontrivial idempotent matrix. Then, the following holds:

(i)Matrix$C-E$be a unit$X_3(\mathbb{Z})$wheredet$(C-E)=1$if and only if
$i-j=1$and$\delta=-1$or$X^2-(1+4\delta)Y^2=4(v+y)^2(2r+1)^2(\delta^2+2\delta+2)$
where
$X=(2r+1)[-(1+4\delta)t+(2\delta+3)(v+y)]$
and$Y=2(v+y)^2s+(2r^2+2r+1+2\delta)t-(2\delta+3)(v+y)$,
or
$i-j=-1$and$\delta=1$or$X^2-(1+4\delta)Y^2=4(v+y)^2(2r+1)^2\delta(\delta-2)$
where
$X=(2r+1)[-(1+4\delta)t+(2\delta-1)(v+y)]$
and$Y=2(v+y)^2s+(2r^2+2r+1+2\delta)t-(2\delta-1)(v+y)$.

(ii) Matrix$C-E$be a unit in$X_3(\mathbb{Z})$wheredet$(C-E)=-1$if and only if
$i-j=1$and$\delta=1$or$X^2-(1+4\delta)Y^2=4(v+y)^2(2r+1)^2\delta(\delta-2)$
where
$X=(2r+1)[-(1+4\delta)t+(2\delta-1)(v+y)]$
dan $Y=2(v+y)^2s+(2r^2+2r+1+2\delta)t-(2\delta-1)(v+y)$,
or
$i-j=-1$and$\delta=-1$or$X^2-(1+4\delta)Y^2=4(v+y)^2(2r+1)^2(\delta^2+2\delta+2)$
where
$X=(2r+1)[-(1+4\delta)t+(2\delta+3)(v+y)]$
and$Y=2(v+y)^2s+(2r^2+2r+1+2\delta)t-(2\delta+3)(v+y)$.

Example ofNil-clean matrix which is not clean in a subring$X_3(\mathbb{Z})$of $M_3(\mathbb{Z})$.

One of the example of3×3 matrixover $\mathbb{Z}$ which is not clean is$\begin{bmatrix}3 & 0 & 9\\ 0 & 1 & 0\\ -7 & 0 & -2\end{bmatrix}$. This example is the extensionfor the result of[2]which giveexample of nil-clean matrix which is not cleanin$M_2(\mathbb{Z})$.

$\begin{bmatrix}3 & 0 & 9\\ 0 & 1 & 0\\ -7 & 0 & -2\end{bmatrix}$is nil-clean matrix since

$$\begin{bmatrix}3 & 0 & 9\\ 0 & 1 & 0\\ -7 & 0 & -2\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ -6 & 0 & 1\end{bmatrix}+\begin{bmatrix}3 & 0 & 9\\ 0 & 0 & 0\\ -1 & 0 & -3\end{bmatrix}$$

where$\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ -6 & 0 & 1\end{bmatrix}$is an idempotent matrixsince$\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ -6 & 0 & 1\end{bmatrix}^2=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ -6 & 0 & 1\end{bmatrix}$,
and$\begin{bmatrix}3 & 0 & 9\\ 0 & 0 & 0\\ -1 & 0 & -3\end{bmatrix}$is a nilpotent matrixsince $\begin{bmatrix}3 & 0 & 9\\ 0 & 0 & 0\\ -1 & 0 & -3\end{bmatrix}^2=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$.

$\alpha=-1,\ u=0,\ v=-6,\ \beta=3,\ x=9,\ y=-1$.We obtain$r=2,\ \delta=-57\neq\pm1$.

Pell's equation related toinvertible matrix$C-E$where$j=0$anddet$(C-E)=1$is$X^2+227Y^2=15{,}371{,}300$where$X=3(227t+777)$ (we shall not need Y).

Since$X=227(3t+10)+61$we obtain$X^2=227k+89$for some integer$k$. Because$15{,}371{,}300=67{,}715\cdot227-5$from Pell's equationwe obtain$X^2=227l-5$for some integer l,which means there is no integer solution.

For Pell's equationrelated to invertible matrix$C-E$where$j=0$anddet$(C-E)=-1$is$X^2+227Y^2=16{,}478{,}700$where$X=3(227t+805)$, by similar way, we obtain$X=227(3t+10)+145$,then$X^2=227p+141$for some integer$p$. Because$16{,}478{,}700=72{,}593\cdot227+89$from Pell's equation we obtain$X^2=227q+89$for some integer q.So, there is no integer solution.

So, $\begin{bmatrix}3 & 0 & 9\\ 0 & 1 & 0\\ -7 & 0 & -2\end{bmatrix}$is a nilclean elementwhich is not clean in subring$X_3(\mathbb{Z})$ of $M_3(\mathbb{Z})$.

V. CONCLUSION

It is known by a result of Diesl [3] that every nil-clean ring is clean. However, in general ring, a nil-clean element is not necessarily clean. Previous research on [2] found counter examples in the form of 2×2 matrix over $\mathbb{Z}$. In this article we extend the matrix order of [2] and found other examples of nil-clean elements which are not clean in a certain subring of $M_3(\mathbb{Z})$. Also we obtain sufficient conditions for nil-clean elements to be clean in a certain subring of $M_3(\mathbb{Z})$

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