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A MATHEMATICAL MODEL OF OPTIMAL CONTROL FOR ADDICTIVE BUYING: PREDICTING THE POPULATION BEHAVIOR

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Abstract: This article deals with the construction of an optimal control problem model for addictive buying. In our model, we divided the population into three classes rational buyers, excessive buyers, and addictive buyers. The division of the total population into subgroups according to activity or identifications of consumer's buying behavior was done by multivariate statistical techniques based on real databases and sociological approaches. The future short term addicted population is computed assuming several future economic scenarios. The forward-backward sweep method is developed to solve the optimal control problem models for addictive buying.

Keywords: Addictive buying, Compartmental model, Compulsive buying, Modeling, Optimal Control.

I. INTRODUCTION

Behavior conventionally identified as addictive buying is an increase in shopping by an individual over time. Addiction is often attributed to social circumstances, the incompleteness of information, lack of foresight, inconsistent plans or health effects. While addictive buying is not specifically described as an uncontrolled problematic buying behavior, it has been studied by psychologists and psychiatrists since the early 20th century [1]. For some people, shopping is a way to manage emotions or express their self-identity. But it can bring about adverse consequences. Furthermore, there is an implicit assumption that current consumption is somehow dependent on previous shopping activity. The degree of the addiction to shopping is commonly thought to be bad, meaning the higher it is the less the individual's enjoyment, or welfare, in the long run. Our analysis will make this notion precise. Some cross-sectional studies have been performed to analyze uncontrolled problematic buying behavior [1] However, to the best of our knowledge, there are no studies that predict the prevalence of this pathological behavior in the next few years.

In 2011, I. García and L. Jódar et al. [2], the authors of this work proposed a compartmental model suitable for describing a discrete mathematical model for addictive buying. Identifications of consumers' buying behavior are performed by using multivariate statistical techniques based on real databases and sociological approaches.

In 2016, César M. Silva and Silvério Rosa et al. [3] considered a compartmental model to study the evolution of the number of regular customers and referral customers in some corporations and present some simulation that illustrates the behavior of the model and discusses its applicability.

In 2018, S. Rosa and P. Rebelo et al. [4] considered an optimal control problem for a non-autonomous model of ODEs that describes the evolution of the number of customers in some company. Namely, we study the best marketing strategy and control it in order to achieve the goals we want by solving optimal control governed by different techniques [8].

This paper is organized as follows. In the next section, the formulation of the optimal control problem will be shown, and the optimal controls will be characterized using Pontryagin's Optimality Principle. In the third section, using the characterization of the controls, numerical examples using a forward-backward sweep method will be illustrated, and the results will be discussed. Lastly, conclusions and possible future works will be discussed.

II. Formulation of The Optimal Control Problem

In this section, to build the mathematical model, the population is divided into three sub populations using the Compulsive Buying Scale proposed in [2]

$$
\begin{aligned}
\frac{dN}{dt} &= \mu P(t) - \delta N(t) - \beta_1 \frac{N(t)A(t)}{P(t)} - \beta_2(t)N(t) + \epsilon_2 A(t),\\
\frac{dS}{dt} &= \beta_1 \frac{N(t)A(t)}{P(t)} + \beta_2(t)N(t) - \delta S(t) - \gamma_1 \frac{S(t)A(t)}{P(t)}\\
&\quad - \gamma_2(t)S(t) + \epsilon_1 A(t),\\
\frac{dA}{dt} &= \gamma_1 \frac{S(t)A(t)}{P(t)} + \gamma_2(t)S(t) - \epsilon_2 A(t) - \delta A(t) - \epsilon_1 A(t),
\end{aligned} \tag{1}
$$

with initial conditions $N(0)$, $S(0)$ and $A(0) \geq 0$

where N is rational buyers, S is excessive buyers and A is addictive buyers such that the total population size P at any given time t is given by $P(t) = N(t) + S(t) + A(t)$. The dynamic of the population can be described by the following system of equations (t, time in months). The solutions for the model are established to be bounded, making it possible for us to apply controls.

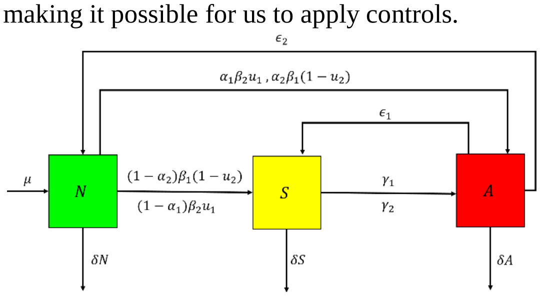

Fig. 1. The compartmental model

Inspired in [5] we assume that the campaign manager can divide its resources in two ways. The first way increases the "spreading rate" where $u_1(t)$ denotes the "word-of-mouth" control signal from the campaign manager. And the last way can directly recruit individuals from the population with rate $u_2(t)$, to be clients (via publicity in mass media). Incentivize shoppers to make recruiting (e.g. discounts, monetary benefits or coupons to current customers who refer their friends to buy a product from the company).

In our work, we assume that all parameters in the mathematical model are positive constants. Based on these assumptions, we can construct a flowchart for the model as shown in Fig. 1.

Hence, the controlled model (state equations), is given as follows.

$$
\begin{aligned}
\frac{dN}{dt} &= \mu P(t) - \delta N(t) - \beta_1(1-u_2(t))\frac{N(t)A(t)}{P(t)}\\
&\quad - \beta_2(t)u_1(t)N(t) + \epsilon_2 A(t),\\
\frac{dS}{dt} &= (1-\alpha_2)\beta_1(1-u_2(t))\frac{N(t)A(t)}{P(t)} - \delta S(t) + \epsilon_1 A(t)\\
&\quad - \gamma_2(t)S(t) - \frac{\gamma_1 S(t)A(t)}{P(t)} + (1-\alpha_1)\beta_2(t)u_1(t)N(t),\\
\frac{dA}{dt} &= \gamma_1 \frac{S(t)A(t)}{P(t)} + \gamma_2(t)S(t) - \epsilon_2 A(t) - \delta A(t) - \epsilon_1 A(t)\\
&\quad + \alpha_1\beta_2(t)u_1(t)N(t) + \alpha_2\beta_1(1-u_2(t))\frac{N(t)A(t)}{P(t)},
\end{aligned} \tag{2}
$$

with initial conditions $N(0), S(0), A(0) \geq 0$, the function u_1, u_2 will be taken in the space L^∞ functions such that $u_1 \in [0, u_{1\max}]$ and $u_2 \in [0, u_{2\max}]$. The goals, which are to minimize the population of the rational buyer's equity at the final time t and to minimize the cost induced from applying both controls, will be realized by minimizing the objective functional

$$J\ (N, u_1, u_2) = \int_0^{t_f} \left(k_1 N(t) + k_2 u_1(t)^2 + k_3 u_2(t)^2\right) dt. \tag{3}$$

In summary, the formulated optimal control problem can be written as

$$\text{minimize } \mathcal{J}(N, u_1, u_2) \quad \text{subject to (2).}$$

All paragraphs must be indented as well as justified, i.e. both left-justified and right-justified.

A. *Parameters of the model*

- μ birth rate in Thailand.
- δ death rate in Thailand.
- β_1 transmission rate due to social contact with addictive buyers (rational buyers → excessive buyers).
- $\beta_2(t)$ rate at which a rational buyer transits to the excessive buyer's sub population. We consider that this parameter is related to economic conditions in Fig. 2.
- γ_1 transmission rate due to social contact with addictive buyers (excessive buyers → addictive buyers).
- $\gamma_2(t)$ rate at which an excessive buyer transits to the addictive buyer's sub population. We consider that this parameter is also related to economic conditions in Fig. 2.

- ε_1 rate at which an addictive buyer reduces his/her addictive behavior himself/herself and becomes an excessive buyer.
- ε_2 rate at which an addictive buyer goes to therapy and becomes a rational buyer.
- α is the percentage of rational buyers among the new buyers.

III. Characterization of The Optimal Controls

In this section, to characterize the optimal controls of the problem (2), we employ Pontryagin's Optimality Principle (POP) [6].The necessary conditions for optimal control are satisfied with the POP. We can generate the necessary conditions from the Hamiltonian equation. Based on that principle, the system of equations (2) and (3) is converted to the Hamiltonian given by

$$\mathcal{H} = k_1 N + k_2 u_1^2 + k_3 u_2^2 + \sum_{i=0}^{3} \lambda_i Y_i \,, \tag{4}$$

where Y_i symbolizes the right-hand side of the differential equation (2) of the i^{th} is state variable. More notably, the state variable is analyzed. Assume $\dot{x}_i$ are state equations containing $\dot{N}, \dot{S}$ and $\dot{A}$, the state equation are obtained by $\dot{x}_i = \partial \mathcal{H} / \partial \lambda_i,\ i = 1, 2, 3$

$$\begin{aligned}
\dot{N} &= mP - dN - \beta_1(1-u_2)\frac{NA}{P} - \beta_2 u_1 N + \varepsilon_2 A,\\
\dot{S} &= (1-\alpha_2)\beta_1(1-u2)\frac{NA}{P} + (1-\alpha_1)\beta_2 u_1 N - dS\\
&\quad - \gamma_1\frac{SA}{P} - \gamma_2 S + \varepsilon_1 A,\\
\dot{A} &= \gamma_1\frac{SA}{P} + \gamma_2 S - \varepsilon_2 A - dA - \varepsilon_1 A + \alpha_1\beta_2 u_1 N\\
&\quad + \alpha_2\beta_1(1-u_2)\frac{NA}{P}.
\end{aligned} \tag{5}$$

Furthermore, adjoint equations are $\dot{\lambda}_i = -\partial \mathcal{H} / \partial x_i$ and follow transversality conditions (boundary conditions) are

$$\lambda_i(T) = 0,\ i = 1, 2, 3, \tag{6}$$

where T is the end of the time period. Thus,

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial \mathcal{H}}{\partial N}\\
&= -k_1 - \lambda_1\left[m - d - \beta_2 u_1 + \frac{\beta_1(u_2-1)A(S+A)}{P^2}\right]\\
&\quad - \lambda_2\left[(1-\alpha_1)\beta_2 u_1 + \frac{\gamma_1 SA + (\alpha_2-1)\beta_1(u_2-1)A(S+A)}{P^2}\right]\\
&\quad - \lambda_3\left[\alpha_1\beta_2 u_1 - \frac{\gamma_1 SA}{P^2} - \frac{\alpha_2\beta_1(u_2-1)A(S+A)}{P^2}\right]\\
\dot{\lambda}_2 &= -\frac{\partial \mathcal{H}}{\partial S}\\
&= -\lambda_1\left[m + \frac{(1-u_2)\beta_1 NA}{P^2}\right] + \lambda_2\gamma_2\\
&\quad + \lambda_2\left[\frac{(1-\alpha_2)\beta_1(1-u_2)NA + \gamma_1 A(N+A)}{P^2} + d\right]\\
&\quad - \lambda_3\left[\frac{\gamma_1 A(N+A) - \alpha_2\beta_1(1-u_2)NA}{P^2} + \gamma_2\right]\\
\dot{\lambda}_3 &= -\frac{\partial \mathcal{H}}{\partial A}\\
&= -\lambda_1\left[m + \frac{(u_2-1)\beta_1 N(N+S)}{P^2} + \varepsilon_2\right]\\
&\quad - \lambda_2\left[\frac{(\alpha_2-1)\beta_1(u_2-1)N(N+S) - \gamma_1 S(N+S)}{P^2} + \varepsilon_1\right]\\
&\quad - \lambda_3\left[\frac{\gamma_1 S(N+S) - \alpha_2\beta_1(u_2-1)N(N+S)}{P^2} - \varepsilon_2 - d - \varepsilon_1\right]
\end{aligned} \tag{7}$$

The optimality conditions are obtain by

$$\frac{\partial \mathcal{H}}{\partial u_i} = 0,\ i = 1, 2\,.$$

Then,

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial u_1} &= 2k_2 u_1 - \lambda_1\beta_2 N + \lambda_2(1-\alpha_1)\beta_2 N + \lambda_3\alpha_1\beta_2 N = 0,\\
\frac{\partial \mathcal{H}}{\partial u_2} &= 2k_3 u_2 + \frac{\lambda_1\beta_1 NA - \lambda_2(1-\alpha_2)\beta_1 NA - \lambda_3\alpha_2\beta_1 NA}{P} = 0.
\end{aligned} \tag{8}$$

To optimality conditions u_1 and u_2, we have

$$\begin{aligned}
u_1 &= \frac{[(\lambda_2-\lambda_3)\alpha_1 + \lambda_1 - \lambda_2]\beta_2 N}{2k_2},\\
u_2 &= -\frac{[(\lambda_2-\lambda_3)\alpha_2 + \lambda_1 - \lambda_2]\beta_1 NA}{2Pk_3}.
\end{aligned} \tag{9}$$

The characterization of the optimal control u_1^* and u_2^* that minimize J (N, u_1, u_2) are presented in the compact form as:

$$\begin{aligned}
u_1^* &= \min\left\{\max\left\{0, \frac{[(\lambda_2-\lambda_3)\alpha_1 + \lambda_1 - \lambda_2]\beta_2 N}{2k_2}\right\}, u_{1\max}\right\},\\
u_2^* &= \min\left\{\max\left\{0, -\frac{[(\lambda_2-\lambda_3)\alpha_2 + \lambda_1 - \lambda_2]\beta_1 NA}{2Pk_3}\right\}, u_{2\max}\right\}.
\end{aligned} \tag{10}$$

After that we set the optimality system

$$\begin{aligned}
\dot{N} &= mP - dN - \beta_1(1-u_2^*)\frac{NA}{P} - \beta_2 u_1^* N + \varepsilon_2 A,\\
\dot{S} &= (1-\alpha_2)\beta_1(1-u_2^*)\frac{NA}{P} + (1-\alpha_1)\beta_2 u_1^* N - dS\\
&\quad - \gamma_1\frac{SA}{P} - \gamma_2 S + \varepsilon_1 A,\\
\dot{A} &= \gamma_1\frac{SA}{P} + \gamma_2 S - \varepsilon_2 A - dA - \varepsilon_1 A + \alpha_1\beta_2 u_1^* N\\
&\quad + \alpha_2\beta_1(1-u_2^*)\frac{NA}{P},
\end{aligned}$$

$$
\begin{aligned}
\dot{\lambda}_1 = & -k_1 - \lambda_1\left[\mu - \delta - b_2 u_1^* + \frac{b_1(u_2^*-1)A(S+A)}{P^2}\right] \\
& - \lambda_2\left[(1-a_1)b_2u_1^* + \frac{g_1SA + (a_2-1)b_1(u_2^*-1)A(S+A)}{P^2}\right] \\
& - \lambda_3\left[a_1b_2u_1^* - \frac{g_1SA}{P^2} - \frac{a_2b_1(u_2^*-1)A(S+A)}{P^2}\right] \\
\dot{\lambda}_2 = & -\lambda_1\left[\mu + \frac{(1-u_2^*)b_1NA}{P^2}\right] + \lambda_2 g_2 \\
& + \lambda_2\left[\frac{(1-a_2)b_1(1-u_2^*)NA + g_1A(N+A)}{P^2} + \delta\right] \\
& - \lambda_3\left[\frac{g_1A(N+A) - a_2b_1(1-u_2^*)NA}{P^2} + g_2\right] \\
\dot{\lambda}_3 = & -\lambda_1\left[\mu + \frac{(u_2^*-1)b_1N(N+S)}{P^2} + \epsilon_2\right] \\
& - \lambda_2\left[\frac{(a_2-1)b_1(u_2^*-1)N(N+S) - g_1S(N+S)}{P^2} + \epsilon_1\right] \\
& - \lambda_3\left[\frac{g_1S(N+S) - a_2b_1(u_2^*-1)N(N+S)}{P^2} - \epsilon_2 - \delta - \epsilon_1\right]
\end{aligned} \tag{11}
$$

The optimality system (11) was solved to reach the optimality control by the forward and backward sweep method that is carried out with the Runge-Kutta fourth-order scheme.

IV. Numerical Simulations

In this section, the optimal control problem (11) with the parameter values in Table-I is numerically solved using a Runge–Kutta fourth-order iterative method.

A. Parameter estimation

Table- I: Parameter of the model with numerical value [2]

Parameter	Value	Units
μ	0.000833	month^{-1}
δ	0.000666	month^{-1}
b_1	0.002453	
$b_2(t)$	$z(t)*(0.25*0.02)$	month^{-1}
γ_1	0.0048	
$g_2(t)$	$z(t)*(0.25*0.02)+k$	month^{-1}
e_1	0.002	month^{-1}
e_2	0.000634	month^{-1}
a_1	0.05	
a_2	0.9	

The function $z(t)$ represents the proportion of the population who have an optimistic opinion related to the economic situation in a month (t) and a constant k.

The function $z(t)$ is estimated by fitting the Consumer Confidence Index (CCI) presented by the TRADING ECONOMICS [7] as Fig. 2.

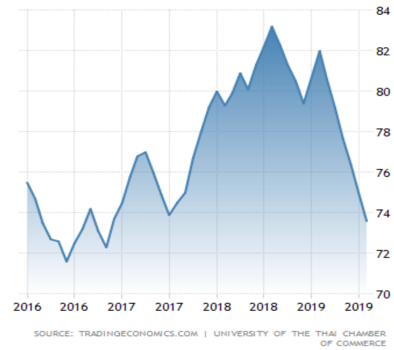

Fig. 2. Consumer Confidence Index (Thailand)

B. Simulation

Here, the first-order necessary conditions derived from the previous section are used to illustrate several scenarios numerically, i.e., with varied values of certain chosen parameters.

Case 1: We assume that the maximum rate of the "word-of-mouth" control signal from the campaign manager is $u_{1\max} = 0.15$. Directly recruit individuals from the population is $u_{2\max} = 1.0$. The terminal time is $t_f = 60$ months. The initial conditions are the following:

$$N_0 = 0.4897,\ S_0 = 0.4018,\ A_0 = 0.1085.$$

We consider that the weight values are

$$k_1 = 1,\ k_2 = 1.5,\ k_3 = 1.25$$

The expression for $z(t)$ by Fig. 2 is:

$$
z(t) = \begin{cases}
0.0005t^5 - 0.018t^4 + 0.2139t^3 - 0.9523t^2 \\
\qquad + 0.6662t + 75.644, & 0 < t \le 13, \\
-0.0075t^3 + 0.5292t^2 \\
\qquad + 11.733t + 158.23, & 13 < t \le 29, \\
-0.00604t^2 + 3.9083t + 18.897, & t > 29,
\end{cases}
$$

where $t = 0$ (*Jan* 2016), $t = 1$ (*Feb* 2016),...

We input the Consumer Confidence Index (CCI) from Fig. 2 (*Jan* 2016 *to Aug* 2019) that represented by the function $\zeta(t)$ in to our control model.

The numerical results shows that the numbers of addictive buyers (A) rapidly grow after control about 40% and the numbers of excessive buyers (S) slightly increase in Fig.3. In Fig.4 shows the numbers of rational buyers rapidly decrease.

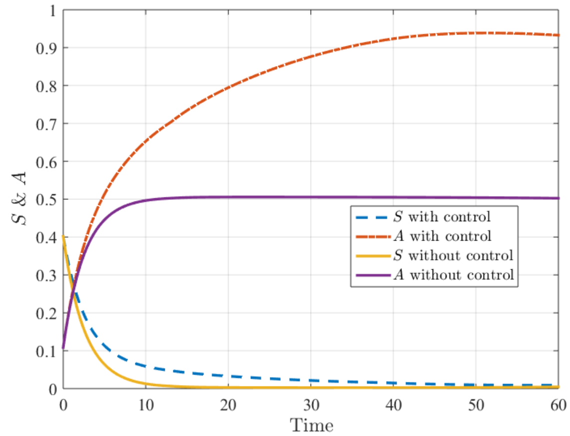

Fig. 3. Evolution of S and A.

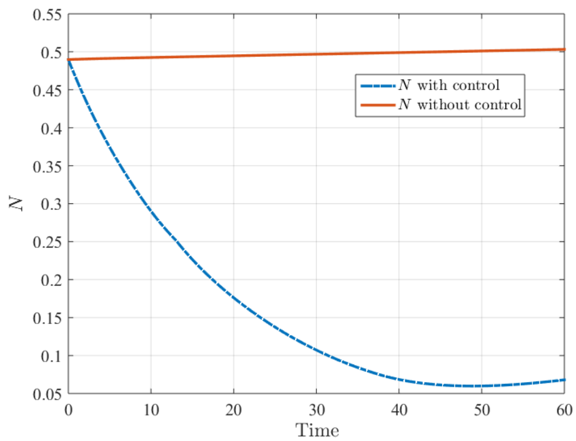

Fig. 4. Evolution of N.

In Fig. 5 and Fig. 6, the first control (u_1) is maximized forwards, $0 < t \le 40$. The second control (u_2) is maximized forwards, $0 < t \le 25$.

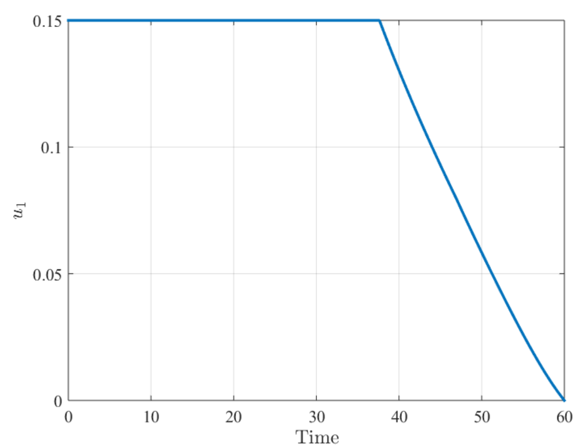

Fig. 5. Variation of the optimal control u_1^*.

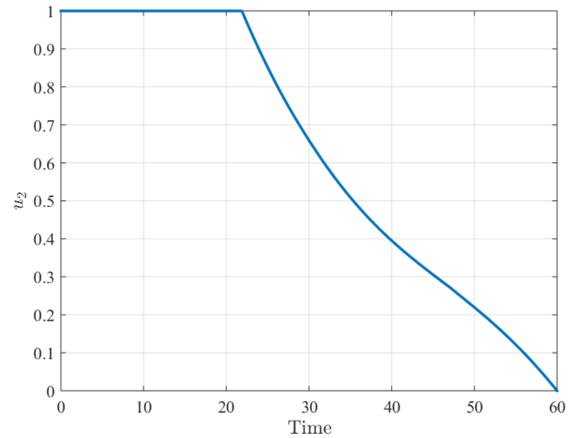

Fig. 6. Variation of the optimal control u_2^*.

Case 2: We assume that the terminal time is $t_f = 25$ months during (Jan 2018 to Aug 2019). We use the expression for the function $\zeta(t)$ by Fig.2. is:

$$z(t) = -0.00006t^5 + 0.0029t^4 - 0.0562t^3 + 0.4419t^2 - 1.0055t + 80.332,$$

where $t \ge 0$, $t = 0$ (Jan 2018), $t = 1$ (Feb 2018),...

In Fig.7 with optimal control, the numerical results show that the numbers of addictive buyers grow vigorously after control (about 30%) and the numbers of excessive buyers slightly increase. In Fig.8 shows that the numbers of rational buyers rapidly decrease.

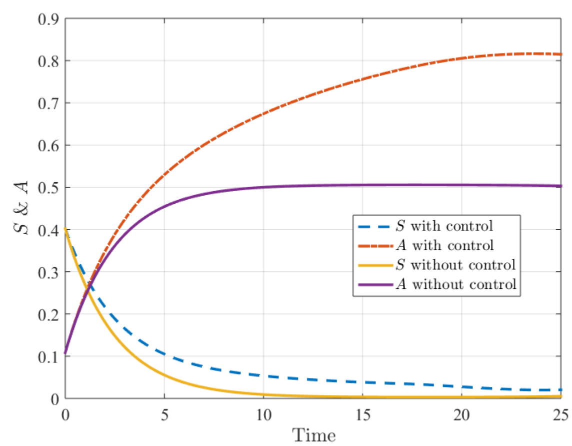

Fig. 7. Evolution of S and A.

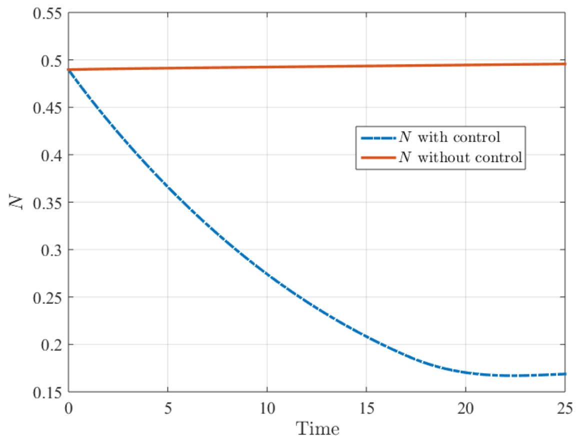

Fig. 8. Evolution of N.

In Figs.9 and 10, the first control (u_1) is maximized forwards, $0 < t \leq 18$. The second control (u_2) is maximized forwards, $0 < t \leq 11$.

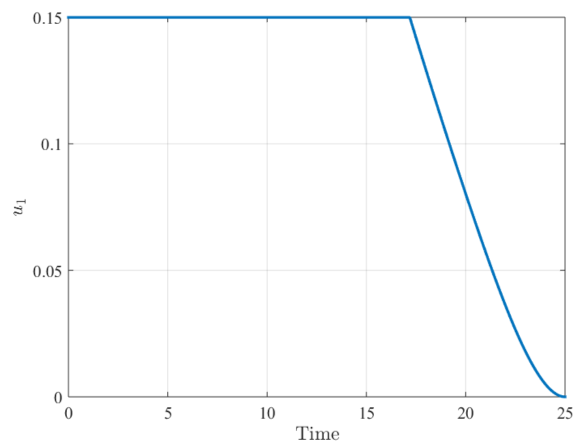

Fig. 9. Variation of the optimal control u_1^*.

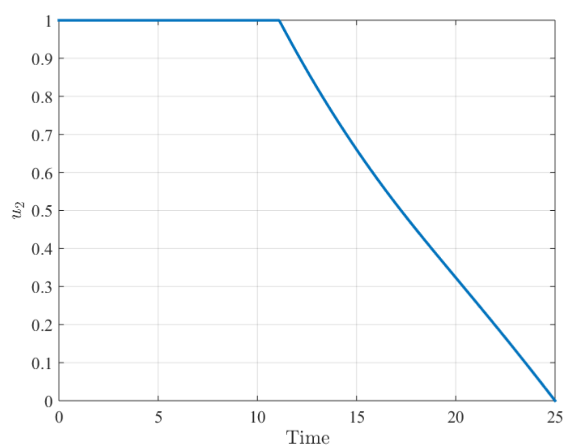

Fig. 10. Variation of the optimal control u_2^*.

V. Conclusion

In this work, we show an approximation of Thailand's addictive buying prevalence for 2016 to 2019 and 2018 to 2019 and some predictions for the next few years. The optimal control mathematical model presents allows us to obtain a first approximation of the population behavior in Thailand. According to the numerical simulations, we can affirm that the addictive buying prevalence rate in Thailand is increasing. We show how mathematical models can represent as a tool for understanding social extensive. Both of the optimal controls are characterized using the Pontryagin's Optimality Principle first-order necessary conditions which are solved. With simulations done by varying, the different scenarios of addictive buyers by using cost-effective control programs was done through a forward-backward sweep method and the optimality system derived and use the necessary conditions. Maximization of the addictive buyers effectively reduces both the rational buyers and excessive buyers by applying both controls in turn.

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