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CHEBYSHEV WAVELET SOLUTIONS FOR TIME-FRACTIONAL INTEGRO PARTIAL DIFFERENTIAL EQUATION AND ITS APPLICATION TO BEAM PROBLEMS

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Abstract: The main objectives of this works are to present a Chebyshev wavelet method to solve approximately analytical solutions which it can apply to the beam problem. The analytical solutions of this problem can be written as Chebyshev wavelet series that can compute the unknown coefficient of Chebyshev wavelet solutions with nonlinear algebraic system. With our numerical results, the Chebyshev wavelet technique is simple and powerful method for calculating any beam problems. The validity and accuracy of our method have been shown through analytical results, absolute error and absolute residue error. Additionally, it is appropriate for solving some fractional order of Caputo fractional in nonlinear time-fractional integro partial differential equations.

Keywords: Chebyshev wavelet method, Caputo time-fractional derivative, Caputo time-fractional integro partial differential equations, beam problem, Chebyshev wavelet solutions.

I. INTRODUCTION

Beam structures are one of the most used elements in structural engineering and it consists of a core that serves to support the vertical weight taken into the support base. The forces that act the beam can produce bending moment and shear forces along the beam which can cause strains, defections, and internal stress. The problem of beams can determine a horizontal structure which has a load point along the length of the beam, causing vertical shear forces. The beam is used for resistance against vertical shear strength and bending moment [1–3].

In this paper, we focus on the initial-boundary value problems for the nonlinear Caputo time-fractional integro partial differential equation which generalizes partial differential equation for beam application of Woinowsky-Krieger [4]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\varepsilon\frac{\partial^4 u(x,t)}{\partial x^4}-\beta\frac{\partial^2 u(x,t)}{\partial x^2} -\kappa\frac{\partial^2 u(x,t)}{\partial x^2}\int_0^L\left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx = f(x,t) \tag{1}$$

where the fractional order $\alpha\in(1,2]$, the constants ε, β and κ are positive, with the initial condition

$$u(x,0) = 0, \tag{2}$$

and the boundary conditions

$$u(0,t)=\frac{\partial^2 u(0,t)}{\partial x^2}=u(L,t)=\frac{\partial^2 u(L,t)}{\partial x^2}=0. \quad (3)$$

II. PRELIMINARIES

In this section, some definitions of the Caputo fractional derivative and properties of Chebyshev polynomials will be introduced.

Definition 1: The Caputo fractional derivative of $u(x)\in AC^n[a,b]$ is defined by [5]

$$D_a^\alpha u(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{u^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}}d\tau, \quad (4)$$

where the order $\alpha\in \mathrm{R}^+$ and $n=\lceil\alpha\rceil$ which is the smallest integer greater than or equal to α. The gamma function $\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt$. Under natural condition on the function $u(x)$, if $\alpha=0$, then $D_a^\alpha u(x)=u(x)$ and if $\alpha\to n$, then $D_a^\alpha u(x)=\frac{d^n u(x)}{dx^n}$. It is obvious that the Caputo fractional derivative is a linear operator similar to integer order differential operators and some properties of Caputo fractional derivatives are as follows [5].

$$D_a^\alpha C=0, \qquad \text{where } C \text{ is a constant} \quad (5)$$

$$D_a^\alpha x^\beta=\begin{cases}0, & \beta<\lceil\alpha\rceil,\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, & \beta\ge\lceil\alpha\rceil.\end{cases} \quad (6)$$

A. *Chebyshev wavelet method*

By a definition, Chebyshev wavelets consist of a family of functions that are coming from dilation and translation of a Chebyshev function named a mother wavelet, which n as a dilation parameter and m as translation parameter vary continuously. The following family of continuous Chebyshev wavelets [6, 7] is defined on the interval $[0,1)$ by

$$\psi_{n,m}(t)=\begin{cases}2^{k/2}\bar{T}_m(2^k t-2n+1), & \frac{n-1}{2^{k-1}}\le t<\frac{n}{2^{k-1}},\\ 0, & \text{otherwise,}\end{cases} \quad (7)$$

where k can be determined as any positive integer and $\bar{T}_m(t)=\sqrt{\frac{2}{\pi}}T_m(t)$, $T_m(t), m=0,1,2,\ldots,M$ are the first kind Chebyshev polynomials which are defined on the interval $[-1,1]$ and satisfy the following recursive formulas

$$T_0(t)=1,$$
$$T_1(t)=2t,$$
$$T_{m+1}(t)=2tT_m(t)-T_{m-1}(t), \qquad m=1,2,3,\ldots,$$

and orthogonal with respect to the weight function $w(t)=\frac{1}{\sqrt{1-t^2}}$.

B. *Operational matrix*

Definition 2: The matrices A and B are given by:

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}_{m\times n}, \quad B=\begin{bmatrix}b_{11}&b_{12}&\cdots&b_{1q}\\b_{21}&b_{22}&\cdots&b_{2q}\\\vdots&\vdots&\ddots&\vdots\\b_{p1}&b_{p2}&\cdots&b_{pq}\end{bmatrix}_{p\times q}.$$

The Kronecker product $A\otimes B$ is the $mp\times nq$ matrix[8, 9]:

$$A\otimes B=\begin{bmatrix}a_{11}B&a_{12}B&\cdots&a_{1n}B\\a_{21}B&a_{22}B&\cdots&a_{2n}B\\\vdots&\vdots&\ddots&\vdots\\a_{m1}B&a_{m2}B&\cdots&a_{mn}B\end{bmatrix}.$$

Here are some Kronecker product properties

- $A\otimes(\alpha B)=\alpha(A\otimes B)$, where α is a scalar.
- $(A+B)\otimes C=(A\otimes C)+(B\otimes C)$.
- $A\otimes(B+C)=(A\otimes B)+(A\otimes C)$.
- $A\otimes(B\otimes C)=(A\otimes B)\otimes C$.
- $(A\otimes B)(C\otimes D)=AC\otimes BD$.

Definition 3: For two $m\times n$ matrices A and B,

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}, \quad B=\begin{bmatrix}b_{11}&b_{12}&\cdots&b_{1n}\\b_{21}&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&\ddots&\vdots\\b_{m1}&b_{m2}&\cdots&b_{mn}\end{bmatrix}.$$

The Hadamard product [10] $A\circ B$ is a matrix of the same dimension as the operands, with elements given by

$$A\circ B=\begin{bmatrix}a_{11}b_{11}&a_{12}b_{12}&\cdots&a_{1n}b_{1n}\\a_{21}b_{21}&a_{22}b_{22}&\cdots&a_{2n}b_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}b_{m1}&a_{m2}b_{m2}&\cdots&a_{mn}b_{mn}\end{bmatrix}$$

where some important properties are given as

- $A\circ B=B\circ A$.
- $A^T\circ B^T=(A\circ B)^T$.
- $(A\circ B)(C\circ D)^T=AC^T\circ BD^T=AD^T\circ BC^T$.
- $C\circ(A+B)=(C\circ A)+(C\circ B)$.
- $\alpha(A\circ B)=(\alpha A)\circ B=A\circ(\alpha B)$.

III. CHEBYSHEV WAVELETS APPROXIMATION

An arbitrary function of two variables $u(x,t)\in L^2(\mathrm{R}\times\mathrm{R})$ defined over $[0,1)\times[0,1)$, can be approximated by Chebyshev wavelets basis as:

$$u(x,t)\approx u^N(x,t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k-1}}\sum_{m'=0}^{M-1}a_{nmn'm'}\psi_{nm}(x)\psi_{n'm'}(t), \quad (8)$$

where the Chebyshev wavelets $\psi_{n,m}(\cdot)$ in Eq. (7). In the other hand, the function $u^N(x,t)$ in Eq. (8) can be rewritten as a finite sum of entries of the spatial matrix

$$u^N(x,t) = \sum_{i=1}^{2^{2k-2}} \sum_{j=1}^{M^2} \Omega_{ij}(x,t), \tag{9}$$

where $\Omega_{ij}(x,t)$ are entries of the Hadamard-Kronecker product matrix $\mathbf{A} \circ (\Psi(x) \otimes \Psi(t))$ where the matrices

$$\mathbf{A} = \begin{bmatrix} a_{1010} & \cdots & a_{1(M-1)1(M-1)} \\ a_{1020} & \cdots & a_{1(M-1)2(M-1)} \\ \vdots & \ddots & \vdots \\ a_{2^{k-1}02^{k-1}0} & \cdots & a_{2^{k-1}(M-1)2^{k-1}(M-1)} \end{bmatrix}, \tag{10}$$

$$\Psi(x) \otimes \Psi(t) = \begin{bmatrix} \psi_{10}(x) \otimes \Psi(t) & \dots & \psi_{1(M-1)}(x) \otimes \Psi(t) \\ \psi_{20}(x) \otimes \Psi(t) & \dots & \psi_{2(M-1)}(x) \otimes \Psi(t) \\ \vdots & \ddots & \vdots \\ \psi_{2^{k-1}0}(x) \otimes \Psi(t) & \cdots & \psi_{2^{k-1}(M-1)}(x) \otimes \Psi(t) \end{bmatrix}$$

and the $2^{k-1} \times M$ matrix

$$\Psi(\cdot) = \begin{bmatrix} \psi_{10}(\cdot) & \psi_{11}(\cdot) & \dots & \psi_{1(M-1)}(\cdot) \\ \psi_{20}(\cdot) & \psi_{21}(\cdot) & \dots & \psi_{2(M-1)}(\cdot) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{2^{k-1}0}(\cdot) & \psi_{2^{k-1}1}(\cdot) & \cdots & \psi_{2^{k-1}(M-1)}(\cdot) \end{bmatrix}. \tag{11}$$

The h-order derivative of Chebyshev wavelets matrix is obtained by $(h = 1, 2, \dots)$

$$D^h\Psi(\cdot) = \begin{bmatrix} D^h\psi_{10}(\cdot) & D^h\psi_{11}(\cdot) & \dots & D^h\psi_{1M-1}(\cdot) \\ D^h\psi_{20}(\cdot) & D^h\psi_{21}(\cdot) & \dots & D^h\psi_{2M-1}(\cdot) \\ \vdots & \vdots & \ddots & \vdots \\ D^h\psi_{2^{k-1}0}(\cdot) & D^h\psi_{2^{k-1}1}(\cdot) & \cdots & D^h\psi_{2^{k-1}M-1}(\cdot) \end{bmatrix},$$

and the Caputo fractional order α derivative of Chebyshev wavelets matrix is given by $(1 < \alpha \le 2)$ in Eq. (6)

$$D^\alpha\Psi(\cdot) = \begin{bmatrix} D_a^\alpha\psi_{10}(\cdot) & D_a^\alpha\psi_{11}(\cdot) & \dots & D_a^\alpha\psi_{1M-1}(\cdot) \\ D_a^\alpha\psi_{20}(\cdot) & D_a^\alpha\psi_{21}(\cdot) & \dots & D_a^\alpha\psi_{2M-1}(\cdot) \\ \vdots & \vdots & \ddots & \vdots \\ D_a^\alpha\psi_{2^{k-1}0}(\cdot) & D_a^\alpha\psi_{2^{k-1}1}(\cdot) & \cdots & D_a^\alpha\psi_{2^{k-1}M-1}(\cdot) \end{bmatrix}.$$

IV. CHEBYSHEV WAVELET SOLUTIONS FOR FPDES

Consider the nonlinear Caputo time-fractional integro partial differential equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \grave{o}\frac{\partial^4 u(x,t)}{\partial x^4} - \beta\frac{\partial^2 u(x,t)}{\partial x^2} - \kappa\frac{\partial^2 u(x,t)}{\partial x^2}\int_0^L \left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx = f(x,t) \tag{12}$$

where $\alpha \in (1,2]$, with the initial condition

$$u(x,0) = 0, \tag{13}$$

and the boundary conditions

$$u(0,t) = \frac{\partial^2 u(0,t))}{\partial x^2} = u(L,t) = \frac{\partial^2 u(L,t))}{\partial x^2} = 0. \tag{14}$$

We assume that the Chebyshev wavelet solution can be written as

$$u^N(x,t) = \sum_{i=1}^{2^{2k-2}} \sum_{j=1}^{M^2} \Omega_{ij}, \tag{15}$$

where Ω_{ij} are entries of the matrix

$$\bar{\Omega} = \mathbf{A}\circ(\Psi(x) \otimes \Psi(t)), \tag{16}$$

and the unknown coefficient matrix $\mathbf{A}$ in Eq. (10) can be determined and the matrix $\Psi(\cdot)$ in Eq. (11).

The matrix $\bar{\Omega}$ can take Caputo fractional derivative with respect to t as

$$\frac{\partial^\alpha \bar{\Omega}}{\partial t^\alpha} = \mathbf{A}\circ(\Psi(x) \otimes D^\alpha\Psi(t)), \tag{17}$$

and also take derivative order n as

$$\frac{\partial^n \bar{\Omega}}{\partial x^n} = \mathbf{A}\circ(D^n\Psi(x) \otimes \Psi(t)), \quad n = 1, 2 \text{ or } 4. \tag{18}$$

Substituting Eqs. (17) and (18) into Eq. (12), it obtains the matrix equation as

$$\begin{aligned} &\mathbf{A} \circ (\Psi(x) \otimes D^\alpha\Psi(t)) + \grave{o}\Phi \circ \mathbf{A} \circ (D^4\Psi(x) \otimes \Psi(t)) \\ &-\beta\Phi \circ \mathbf{A} \circ (D^2\Psi(x) \otimes \Psi(t)) - \kappa\Phi \circ \mathbf{A} \circ (D^2\Psi(x) \otimes \Psi(t)) \\ &\int_0^L \left|\mathbf{A} \circ (D\Psi(x) \otimes \Psi(t))\right|^2 dx = f(x,t)\Phi, \end{aligned} \tag{19}$$

where the matrix Φ is

$$\Phi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{2^{2k-2} \times M^2}.$$

The matrices of the initial and boundary conditions in Eq. (13) and Eq. (14) are then

$$\mathbf{A} \circ (\Psi(x) \otimes \Psi(0)) = \bar{\mathbf{0}}, \tag{20}$$

$$\mathbf{A} \circ (\Psi(0) \otimes \Psi(t)) = \bar{\mathbf{0}}, \tag{21}$$

$$\mathbf{A} \circ (D^2\Psi(0) \otimes \Psi(t)) = \bar{\mathbf{0}}, \tag{22}$$

$$\mathbf{A}\circ(\Psi(L) \otimes \Psi(t)) = \bar{\mathbf{0}}, \tag{23}$$

$$\mathbf{A}\circ(D^2\Psi(L) \otimes \Psi(t)) = \bar{\mathbf{0}}. \tag{24}$$

Next, the collocation points of time and space are defined as

$$x_i = \frac{2i-1}{2m}, \quad t_i = \frac{2i-1}{2m}, \quad i = 1, 2, \dots, 2^{k-1}M. \tag{25}$$

Picking $x = 0$ and $t = 0, t_2, t_3, \dots, t_{2^{k-1}M}$ and substituting into Eq. (21) we have

$$\mathbf{A}\circ\left(\Psi(0)\otimes\Psi(0)\right)=\bar{\mathbf{0}},$$
$$\mathbf{A}\circ\left(\Psi(0)\otimes\Psi(t_2)\right)=\bar{\mathbf{0}},$$
$$\mathbf{A}\circ\left(\Psi(0)\otimes\Psi(t_3)\right)=\bar{\mathbf{0}},$$
$$\vdots$$
$$\mathbf{A}\circ\left(\Psi(0)\otimes\Psi(t_{2^{k-1}M})\right)=\bar{\mathbf{0}}.$$

Picking $x=0$ and $t=0,t_2,t_3,\ldots,t_{2^{k-1}M}$ and substituting into Eq. (22) we have

$$\mathbf{A}\circ\left(D^2\Psi(0)\otimes\Psi(0)\right)=\bar{\mathbf{0}}$$
$$\mathbf{A}\circ\left(D^2\Psi(0)\otimes\Psi(t_2)\right)=\bar{\mathbf{0}}$$
$$\mathbf{A}\circ\left(D^2\Psi(0)\otimes\Psi(t_3)\right)=\bar{\mathbf{0}}$$
$$\vdots$$
$$\mathbf{A}\circ\left(D^2\Psi(0)\otimes\Psi(t_{2^{k-1}M})\right)=\bar{\mathbf{0}}.$$

Picking $x=x_3$ and $t=0,t_2,t_3,\ldots,t_{2^{k-1}M}$, and substituting into Eqs. (19) and (20) we have

$$\mathbf{A}\circ\left(\Psi(x_3)\otimes\Psi(0)\right)=\bar{\mathbf{0}}$$
$$\mathbf{A}\circ\left(\Psi(x_3)\otimes D^{\alpha}\Psi(t_2)\right)$$
$$+\epsilon\Phi\circ\mathbf{A}\circ\left(D^4\Psi(x_3)\otimes\Psi(t_2)\right)$$
$$-\beta\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_3)\otimes\Psi(t_2)\right)$$
$$-\kappa\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_3)\otimes\Psi(t_2)\right)$$
$$\int_0^L\left|\mathbf{A}\circ\left(D\Psi(x_3)\otimes\Psi(t_2)\right)\right|^2dx=f(x_3,t_2)\Phi,$$
$$\mathbf{A}\circ\left(\Psi(x_3)\otimes D^{\alpha}\Psi(t_3)\right)$$
$$+\epsilon\Phi\circ\mathbf{A}\circ\left(D^4\Psi(x_3)\otimes\Psi(t_3)\right)$$
$$-\beta\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_3)\otimes\Psi(t_3)\right)$$
$$-\kappa\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_3)\otimes\Psi(t_3)\right)$$
$$\int_0^L\left|\mathbf{A}\circ\left(D\Psi(x_3)\otimes\Psi(t_3)\right)\right|^2dx=f(x_3,t_3)\Phi,$$
$$\vdots$$
$$\mathbf{A}\circ\left(\Psi(x_3)\otimes D^{\alpha}\Psi(t_{2^{k-1}M})\right)$$
$$+\epsilon\Phi\circ\mathbf{A}\circ\left(D^4\Psi(x_3)\otimes\Psi(t_{2^{k-1}M})\right)$$
$$-\beta\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_3)\otimes\Psi(t_{2^{k-1}M})\right)$$
$$-\kappa\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_3)\otimes\Psi(t_{2^{k-1}M})\right)$$
$$\int_0^L\left|\mathbf{A}\circ\left(D\Psi(x_3)\otimes\Psi(t_{2^{k-1}M})\right)\right|^2dx=f(x_3,t_{2^{k-1}M})\Phi.$$
$$\vdots$$

Picking $x=x_\eta$ where $\eta=2^{k-1}M-2$ and $t=0,t_2,t_3,\ldots,t_{2^{k-1}M}$ and substituting into Eqs. (19) and (20) we have

$$\mathbf{A}\circ\left(\Psi(x_\eta)\otimes\Psi(0)\right)=\bar{\mathbf{0}}$$
$$\mathbf{A}\circ\left(\Psi(x_\eta)\otimes D^{\alpha}\Psi(t_2)\right)$$
$$+\epsilon\Phi\circ\mathbf{A}\circ\left(D^4\Psi(x_\eta)\otimes\Psi(t_2)\right)$$
$$-\beta\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_\eta)\otimes\Psi(t_2)\right)$$
$$-\kappa\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_\eta)\otimes\Psi(t_2)\right)$$
$$\int_0^L\left|\mathbf{A}\circ\left(D\Psi(x_\eta)\otimes\Psi(t_2)\right)\right|^2dx=f(x_\eta,t_2)\Phi,$$
$$\mathbf{A}\circ\left(\Psi(x_\eta)\otimes D^{\alpha}\Psi(t_3)\right)$$
$$+\epsilon\Phi\circ\mathbf{A}\circ\left(D^4\Psi(x_\eta)\otimes\Psi(t_3)\right)$$
$$-\beta\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_\eta)\otimes\Psi(t_3)\right)$$
$$-\kappa\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_\eta)\otimes\Psi(t_3)\right)$$
$$\int_0^L\left|\mathbf{A}\circ\left(D\Psi(x_\eta)\otimes\Psi(t_3)\right)\right|^2dx=f(x_\eta,t_3)\Phi,$$
$$\vdots$$
$$\mathbf{A}\circ\left(\Psi(x_\eta)\otimes D^{\alpha}\Psi(t_{2^{k-1}M})\right)$$
$$+\epsilon\Phi\circ\mathbf{A}\circ\left(D^4\Psi(x_\eta)\otimes\Psi(t_{2^{k-1}M})\right)$$
$$-\beta\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_\eta)\otimes\Psi(t_{2^{k-1}M})\right)$$
$$-\kappa\Phi\circ\mathbf{A}\circ\left(D^2\Psi(x_\eta)\otimes\Psi(t_{2^{k-1}M})\right)$$
$$\int_0^L\left|\mathbf{A}\circ\left(D\Psi(x_\eta)\otimes\Psi(t_{2^{k-1}M})\right)\right|^2dx=f(x_\eta,t_{2^{k-1}M})\Phi.$$

Picking $x=L$ and $t=0,t_2,t_3,\ldots,t_{2^{k-1}M}$ and substituting into Eq. (24) we have

$$\mathbf{A}\circ\left(D^2\Psi(L)\otimes\Psi(0)\right)=\bar{\mathbf{0}}$$
$$\mathbf{A}\circ\left(D^2\Psi(L)\otimes\Psi(t_2)\right)=\bar{\mathbf{0}}$$
$$\mathbf{A}\circ\left(D^2\Psi(L)\otimes\Psi(t_3)\right)=\bar{\mathbf{0}}$$
$$\vdots$$
$$\mathbf{A}\circ\left(D^2\Psi(L)\otimes\Psi(t_{2^{k-1}M})\right)=\bar{\mathbf{0}}.$$

Finally, Picking $x=L$ and $t=0,t_2,t_3,\ldots,t_{2^{k-1}M}$ and substituting into Eq. (23) we have

$$\mathbf{A}\circ\left(\Psi(L)\otimes\Psi(0)\right)=\bar{\mathbf{0}},$$
$$\mathbf{A}\circ\left(\Psi(L)\otimes\Psi(t_2)\right)=\bar{\mathbf{0}},$$
$$\mathbf{A}\circ\left(\Psi(L)\otimes\Psi(t_3)\right)=\bar{\mathbf{0}},$$
$$\vdots$$
$$\mathbf{A}\circ\left(\Psi(L)\otimes\Psi(t_{2^{k-1}M})\right)=\bar{\mathbf{0}}.$$

For the system $2^{2k-2}\times M^2$ equations, solving coefficients of the matrix $\mathbf{A}$ must include all initial and boundary conditions in Eqs. (20) – (24) matrices. We next apply Newton's iterative method and Maple program to calculate all coefficients of the matrix $\mathbf{A}$. It provides the known approximate solution $u^N(x,t)$ in Eq. (9) which is the analytical solution of the time-fractional integro PDEs for application to the beam problem.

V. NUMERICAL RESULTS

In this section, we give some examples to present the applicability and preciseness of the proposed method. All numerical computations were operated using the Maple program.

Example 1: Consider the nonlinear integro partial differential equation of the beam problem

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial^4 u(x,t)}{\partial x^4}-\frac{\partial^2 u(x,t)}{\partial x^2} -2\frac{\partial^2 u(x,t)}{\partial x^2}\int_0^1\left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx = f(x,t),$$

$0\le x\le 1,\ 0\le t\le 1,$ with the initial condition

$$u(x,0) = 0,$$

and the boundary conditions

$$u(0,t)=\frac{\partial^2 u(0,t))}{\partial x^2}=u(1,t)=\frac{\partial^2 u(1,t))}{\partial x^2}=0,$$

where $f(x,t)=\pi^4\sin(\pi x)\sin(\pi t)-\pi^2\sin(\pi x)\sin(\pi t)+\pi^2\sin(\pi x)\sin(\pi t)(1-\pi^2\cos(\pi t)^2+\pi^2),$ which the exact solution is $u(x)=\sin(\pi x)\sin(\pi t).$

By applying the Chebyshev wavelet method, the approximate solution with $(k=1, M=8)$ is given by:

$$\begin{aligned} u^N(x,t) &= 3.282\times10^{-11}+1.482x+0.0004x^2-2.443x^3 \\ &+0.019x^4+1.145x^5+0.114x^6-0.425x^7+0.106x^8 \\ &+\left(5.136\times10^{-8}x+1.8\times10^{-19}-5.099\times10^{-7}x^2\right. \\ &+1.206\times10^{-6}x^3-2.382\times10^{-7}x^4-3.486\times10^{-6}x^5 \\ &\left.+6.243\times10^{-6}x^6-4.355\times10^{-6}x^7+1.089\times10^{-6}x^8\right) \\ &(2t-1)+\left(-1.569x-4.571\times10^{-11}-4.231\times10^{-4}x^2\right. \\ &+2.584x^3-0.02x^4-1.211x^5-0.121x^6+0.449x^7 \\ &\left.-0.112x^8\right)\left(8t^2-8t+1\right)+\ldots+\left(2.119\times10^{-5}x\right. \\ &-1.799\times10^{-6}x^2-3.097\times10^{-5}x^3-3.483\times10^{-6}x^4 \\ &+2.106\times10^{-5}x^5-4.178\times10^{-5}x^6-2.436\times10^{-5}x^7 \\ &\left.+6.089\times10^{-7}x^8+1.561\times10^{-16}\right)\left(327t^8-131t^7\right. \\ &+212t^6-180224t^5+84480t^4-21504t^3+2688t^2 \\ &\left.-128t+1\right). \end{aligned}$$

The accuracy of our method can be illustrated by the absolute error $|u_{exact}-u^N|$ in Table-I and the graph that represents the Chebyshev wavelet solution is depicted in Figure 1.

Table- I: Absolute errors of numerical result for **Example 1**.

	t = 0.2	**t = 0.5**	**t = 0.8**
x = 0.1	2.197×10^{-6}	3.321×10^{-6}	2.202×10^{-6}
x = 0.2	4.269×10^{-6}	5.975×10^{-6}	4.286×10^{-6}
x = 0.3	5.544×10^{-6}	7.248×10^{-6}	5.572×10^{-6}
x = 0.4	6.299×10^{-6}	7.891×10^{-6}	6.338×10^{-6}
x = 0.5	6.549×10^{-6}	8.085×10^{-6}	6.591×10^{-6}
x = 0.6	6.299×10^{-6}	7.892×10^{-6}	6.338×10^{-6}
x = 0.7	5.543×10^{-6}	7.249×10^{-6}	5.575×10^{-6}
x = 0.8	4.271×10^{-6}	5.976×10^{-6}	4.287×10^{-6}
x = 0.9	2.198×10^{-6}	3.325×10^{-6}	2.204×10^{-6}

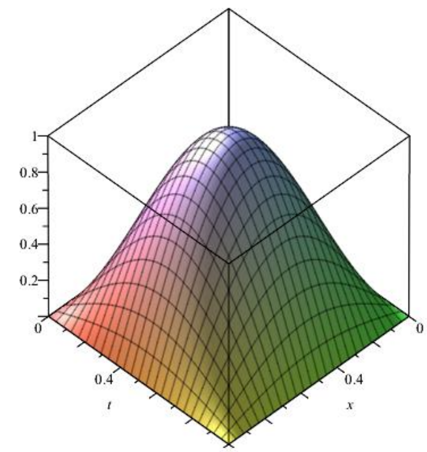


Figure1 Graph of the solution for Example 1.

Example 2: Consider the nonlinear integro partial differential equation of the beam problem

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial^4 u(x,t)}{\partial x^4}-\frac{\partial^2 u(x,t)}{\partial x^2} -2\frac{\partial^2 u(x,t)}{\partial x^2}\int_0^1\left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx = xt,$$

$0\le x\le 1,\ 0\le t\le 1,$ withthe initial condition

$$u(x,0) = 0,$$

and the boundary conditions

$$u(0,t)=\frac{\partial^2 u(0,t))}{\partial x^2}=u(1,t)=\frac{\partial^2 u(1,t))}{\partial x^2}=0.$$

We firstly define an absolute residual error given by

$$E_r = |Lu - f|,$$

Where, the operator $Lu=\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial^4 u(x,t)}{\partial x^4}-\frac{\partial^2 u(x,t)}{\partial x^2}$ $-2\frac{\partial^2 u(x,t)}{\partial x^2}\int_0^1\left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx$ and $f(x,t) = xt.$

By applying the Chebyshev wavelet method, the Chebyshev wavelet solution can be computed as:

$$u^N(x,t) = 1.0\times10^{-12}x + 0.008x^2 - 0.012x^3 + \ldots + 0.059x^8 + \left(1.0\times10^{-12}x + 0.008x^2 - 0.013x^3 + 0.068x^4 + 0.004x^5 + 0.004x^6 + 0.007x^7 + 0.597x^8 - 5.73\times10^{-14}\right)(2t-1) + \ldots + \left(-1.286\times10^{-13}x + 3.7\times10^{-25} + 4.524\times10^{-13}x^2 + 7.96\times10^{-13}x^3 - 5.07\times10^{-12}x^4 + 9.35\times10^{-12}x^5 - 9.76\times10^{-12}x^6 + 5.713\times10^{-12}x^7 - 1.406\times10^{-12}x^8\right)\left(32768t^8 - 131072t^7 + 2129t^6 - 1802t^5 + 844t^4 - 215t^3 + 2688t^2 - 128t + 1\right).$$

The absolute residual errors are reported in Table-II

Table-II: Absolute residual errors of numerical result for **Example** 2.

	***t* = 0.2**	***t* = 0.5**	***t* = 0.8**
***x* = 0.1**	4.695×10^{-6}	1.174×10^{-6}	1.878×10^{-5}
***x* = 0.2**	3.864×10^{-7}	9.668×10^{-7}	1.547×10^{-6}
***x* = 0.3**	2.189×10^{-7}	5.477×10^{-7}	8.768×10^{-6}
***x* = 0.4**	2.916×10^{-7}	7.287×10^{-7}	1.165×10^{-6}
***x* = 0.5**	3.017×10^{-10}	1.06×10^{-10}	1.57×10^{-10}
***x* = 0.6**	6.299×10^{-7}	7.405×10^{-7}	1.624×10^{-6}
***x* = 0.7**	2.268×10^{-7}	5.662×10^{-7}	9.056×10^{-7}
***x* = 0.8**	4.065×10^{-7}	1.016×10^{-6}	4.287×10^{-6}
***x* = 0.9**	5.012×10^{-6}	1.253×10^{-5}	2.005×10^{-5}

and the graph of the solution is shown in Fig. 2. $(k=1, M=8)$.

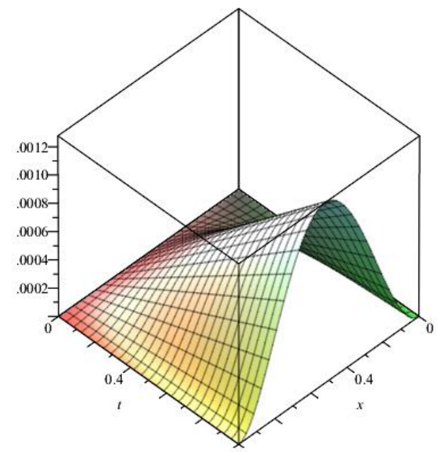


Figure 2 Graph of the solution for Example 2.

Example 3: Consider the nonlinear integro partial differential equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^4 u(x,t)}{\partial x^4} - \frac{\partial^2 u(x,t)}{\partial x^2} - 2\frac{\partial^2 u(x,t)}{\partial x^2}\int_0^1 \left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx = \frac{1}{x+t+1},$$

$0 \le x \le 1,\ 0 \le t \le 1,$ where $1 < \alpha \le 2$ with the initial condition

$$u(x,0) = 0,$$

and the boundary conditions

$$u(0,t) = \frac{\partial^2 u(0,t))}{\partial x^2} = u(1,t) = \frac{\partial^2 u(1,t))}{\partial x^2} = 0.$$

In procedure of Chebyshev wavelet technique, the Chebyshev wavelet solutions with different kinds of fractional order $\alpha = 1.4, 1.6, 1.8$ and 2, respectively can be calculated as

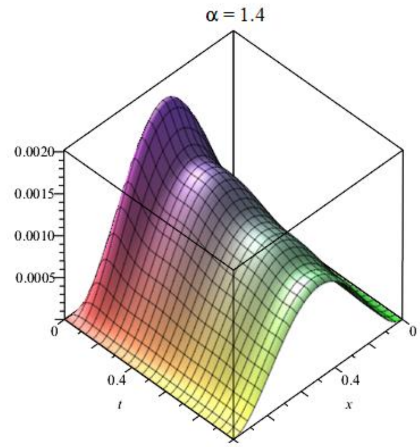
$$u^N_{\alpha=1.4}(x,t) = 2.0\times10^{-12}x + \left(6.0\times10^{-13}x + 0.004x^2 - 0.008x^3 + 0.004x^4 - 0.001x^5 + 0.001x^6 - 0.094x^7 + 0.023x^8 - 2.768\times10^{-14}\right)\left(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + 50t - 1\right) + \left(4.22\times10^{-15} - 0.003x^2 + 0.007x^3 - 0.004x^4 + 0.097x^5 - 0.001x^6 + 0.074x^7 - 0.018x^8\right)\left(2048t^6 - 6144t^5 + 691t^4 - 358t^3 + 840t^2 - 72t + 1\right)\left(1.0\times10^{-12}x - 0.004x^2 + 0.009x^3 - 0.006x^4 + 0.001x^5 - 0.001x^6 + 0.008x^7 - 0.019x^8 - 9.97\times10^{-14}\right)\left(128t^4 - 256t^3 + 160t^2 - 32t + 1\right)$$

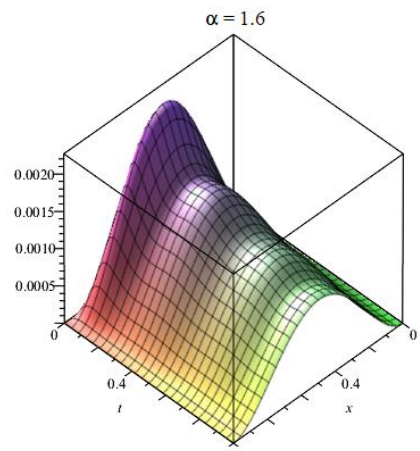
$$u_{\alpha=1.6}^{N}(x,t)=0.022x^2-0.046x^3+0.027x^4-0.006x^5+0.005x^6 -0.003x^7+0.008x^8+\left(-3.0\times10^{-13}x-0.003x^2 +0.006x^3-0.003x^4+0.002x^5-0.004x^6+0.003x^7 -0.001x^8+1.03\times10^{-14}\right)(2t-1)+6.28\times10^{-14} +\left(-5.43\times10^{-15}-0.003x^2+0.007x^3-0.005x^4 +0.001x^5+0.009x^6-0.009x^7+0.002x^8\right)\left(8t^2 -8t+1\right)+\ldots+\left(1.0\times10^{-13}x-0.001x^2+0.003x^3 -0.001x^4+0.003x^5-0.08x^6+0.006x^7-0.015x^8 -8.2\times10^{-15}\right)\left(32768t^8-131072t^7+212992t^6 -1802t^5+8440t^4-2154t^3+2688t^2-128t+1\right)$$

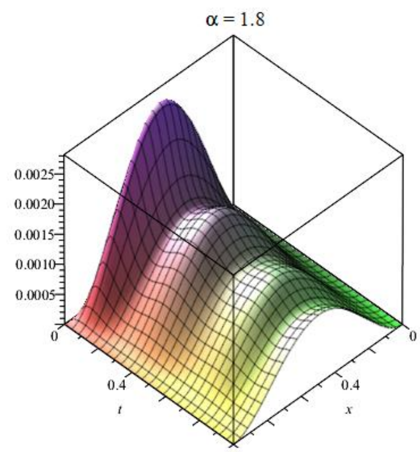
$$u_{\alpha=1.8}^{N}(x,t)=-2.0\times10^{-12}x+0.023x^2-0.048x^3+0.027x^4 -0.006x^5+0.007x^6-0.005x^7+0.001x^8 +\left(-0.01x^2+0.005x^3-0.005x^4+0.007x^5+0.04x^6 -0.004x^7+0.001x^8+3.9\times10^{-15}\right)\left(8t^2-8t+1\right) +\left(0.004x^2-0.008x^3+0.006x^4-0.001x^5-0.001x^6 +0.001x^7-0.019x^8-2.1\times10^{-14}\right)\left(32t^3-48t^2 +18t-1\right)+\ldots+\left(-0.05x^2+0.009x^3-0.003x^4 +0.002x^5-0.007x^6+0.006x^7-0.001x^8 -7.7\times10^{-15}\right)(2t-1)-3.046\times10^{-14}$$

$$u_{\alpha=2}^{N}(x,t)=-2.0\times10^{-12}x+\left(-4.7\times10^{-14}-0.006x^2+0.012x^3 -0.004x^4+0.002x^5-0.007x^6+0.006x^7-0.001x^8\right) \left(2048t^6-6144t^5+6912t^4-3584t^3+840t^2-72t+1\right) +\left(5.3\times10^{-14}+0.011x^2-0.018x^3+0.004x^4 -0.003x^5+0.017x^6-0.014x^7+0.003x^8\right)\left(8192t^7 -28672t^6+39424t^5-26880t^4+9408t^3-1568t^2 +98t-1\right)+\ldots+\left(1.0\times10^{-12}x+4.0\times10^{-14} +0.006x^2-0.012x^3+0.005x^4-0.002x^5+0.006x^6 -0.004x^7+0.001x^8\right)\left(512t^5-1280t^4+1120t^3 -400t^2+50t-1\right).$$

Graphs of the approximate Chebyshev wavelet solutions for order $\alpha=1.4,\ 1.6,\ 1.8$ and 2 are shown in Fig.3 $(k=1, M=8)$.







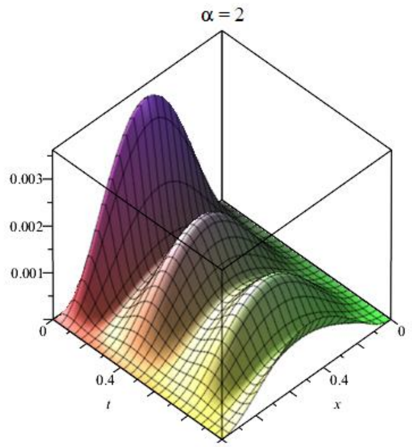


Figure 3 Graphs of the solutions where $\alpha = 1.4$, 1.6, 1.8 and 2

VI. CONCLUSION

In this paper, the Chebyshev wavelet method is applied to solve initial-boundary value problems of the Caputo time-fractional integro-differential partial differential equation which application for a beam problem. This method is simple and a good mathematical method for finding analytical solutions of Caputo time-fractional integro-differential partial differential equation. The validity, accuracy and applicability of our Chebyshev wavelet method have been illustrated through numerical results by showing the absolute errors between an exact solution and Chebyshev wavelet solutions in Table-1 and accuracy and efficiency of our method are reported by the absolute residual errors in Table-2. Moreover, Chebyshev wavelet technique is powerful method for solving Caputo time-fractional nonlinear integro partial differential equations in some varieties of fractional order α of Caputo fractional derivative as shown in Example 3.

VII. ACKNOWLEDGMENT

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