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ON THE APPLICATION OF ELIMINATION IDEAL FOR STATISTICAL MODEL

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Abstract: The set of parameters by which the true density function is realizable by a statistical model is a common set of zero points defined by finite polynomials by applying Hilbert's basis theorem. We calculate the parameterization of this algebraic set by considering the elimination ideal of the ideal defined by the finite polynomials. In this paper, we show the calculation results for recurrence formula by using computer algebra system.

Keywords: statistical model; elimination ideal; Gröbner basis

I. INTRODUCTION

Let the statistical model be a three-layer neural network with $H = 1$input unit, n hidden units, 1 output unit, and let the activation function be defined by the hyperbolic tangent and let the true density function be a three-layer neural network with $H_0 = m$ hidden units.
We consider the analytic set of parameters by which the true density function is realizable by the statistical model.

$$\begin{aligned} W_0 &:= \{w \in R^{2n} | p(y|x,w) = q(y,x)\} \\ &= \{w \in R^{2n} | a_1 \tanh(b_1 x) + \cdots + a_n \tanh(b_n x) \\ &= a_1' \tanh(b_1' x) + \cdots + a_m' \tanh(b_m' x) \end{aligned}$$

Let us define $d = n + m$. Since $\tanh(x)$is an odd function, let b_i be satisfying the condition $b_i \geq 0$. By using the Taylor expansion of the activation function, we obtain

$$g_k(a_1, \dots, a_n, b_1, \dots, b_n, a_1', \dots, a_m', b_1', \dots, b_m') := \sum_{i=1}^{n} a_i b_i^{2k+1} - \sum_{i=1}^{m} a_i' b_i'^{2k+1}.$$

The defining equation is represented as follows:

$$\sum_{k=0}^{\infty} \frac{2^{2k+2}(2^{2k+2}-1)B_{2k+2}x^{2k+1}}{(2k+2)!} g_k$$

where g_k is a function of $a_1, \dots, a_n, b_1, \dots, b_n, a_1', \dots, a_m', b_1', \dots, b_m'$, and $B_n$means denotes the Bernoulli's numbers. Since the set of functions $\{x^{2k+1}\}$ are linearly independent, W_0 is a common zero point defined by infinite polynomials: $W_0 := \{w \in R^{2n} | g_0 = g_1 = \cdots = 0\}$ where $I_k = \langle g_0, g_1, g_2, \dots, g_k \rangle$, then it defines a non decreasing sequence of ideals. In the elimination theory, one of basic strategy is Elimination Theorem [1].

Theorem 1(Elimination Theorem) Let $I \subset k[x_1, \dots, x_n]$ be an ideal and let G be a Gröbner basis of $I$with respect to lex order where $x_1 > x_2 > \cdots > x_n$. Then, for every $0 \leq l \leq n$, the set $G_1 = G \cap k[x_{l+1}, \dots, x_n]$is a Gröbner basis of the lth elimination ideal I_l.

For the elucidation of various phenomena, the result of the calculation of the elimination ideal becomes the key.

II. APPLICATION IN A STATISTICAL MODEL

By using taylor expansion of activation function, the set of parameters that true density function is realizable by a statistical model is a common zero points defined by finite polynomials [2].

Theorem 2 (Hilbert's basis theorem) For an arbitrary ideal I in $R[x_1, x_2, \dots, x_d]$, the ideal is generated by a finite polynomials.

Lemma 1: For 4-variable polynomial $g_k(a,b,c,d)$, let us define an ideal $I_k = \langle g_0, g_1, g_2, \dots, g_k \rangle$. Then this polynomial defines a nondecreasing sequence of ideals. There exists $k$such that $I_1 = I_k$.

Proof: Using Mathematica, we input as the following:

$f_1 = g_0 - ab - cd;$
$f_2 = g_1 - ab^3 - cd^3;$
$f_3 = g_2 - ab^5 - cd^5;$
$GroebnerBasis[\{f_1, f_2, f_3\}, \{a, b, c, d\},$
$MonomialOrder -> Lexicographic]$

The output is Gröbnerbasis. We input the following to calculate the Gröbner basis of the ideal eliminated a, c:

$GroebnerBasis[\{f_1, f_2, f_3\}, \{a, b, c, d\}, \{a, c\},$
$MonomialOrder -> Lexicographic]$

Then the output is as follows: $b^2d^2g_0 - b^2g_1 - d^2g_1 + g_2$. Therefore, $g_2 \in \langle g_0, g_1 \rangle$. We input as the follows:

$f_4 = g_3 - ab^7 - cd^7;$
$GroebnerBasis[\{f_1, f_2, f_3, f_4\}, \{a, b, c, d\},$
$MonomialOrder -> Lexicographic]$

The output is Gröbnerbasis. We input the following to calculate the another Gröbnerbasis of the ideal eliminated a, c:
$GroebnerBasis[\{f_1, f_2, f_3, f_4\}, \{a, b, c, d\}, \{a, c, g_2\},$
$MonomialOrder -> Lexicographic]$

Then the output is as follows:
$b^4d^2g_0 + b^2d^4g_0 - b^4g_1 - b^2d^2g_1 - d^4g_1 + g_3$.
Therefore, $g_3 \in \langle g_0, g_1 \rangle$.
By using Mathematica, similarly, we get: $g_2, g_3 \in \langle g_0, g_1 \rangle$.
Then, the following recurrence formula holds:

$$g_{k+1} = g_1(b^{2k} + d^{2k}) - g_0(b^2d^{2k} + b^{2k}d^2) + (b^2d^2)g_{k-1}.$$

Thus, $\forall k \geq 1$, it holds as follows: $g_k \in \langle g_0, g_1 \rangle$. □

Lemma 2: For 6-variable polynomial $g_k(a,b,c,d,e,f)$, let us define an ideal I_k. Then this polynomial defines a nondecreasing sequence of ideals. There exists k such that $I_2 = I_k$.

Proof: Since Mathematica can't calculate with 6-variables, as the first step, we calculate with a basis of 4-variable polynomials. We input as follows:

$f_1 = g[0] - g_0 - eh;\ f_2 = g[1] - g_1 - eh^3;$
$f_3 = g[2] + b^2d^2g_0 - (b^2 + d^2)g_1 - eh^5;$
$f_4 = g[3] + (b^4d^2 + b^2d^4)g_0 - (b^4 + b^2d^2 + d^4)g_1 - eh^7;$
$GroebnerBasis[\{f_1, f_2, f_3, f_4\}, \{b, d, e, h\},$
$MonomialOrder -> Lexicographic]$

We input the following to calculate the Gröbnerbasis of the ideal eliminated e, g_0, g_1.

$GroebnerBasis[\{f_1, f_2, f_3, f_4\}, \{b, d, h\}, \{e, g_0, g_1\},$
$MonomialOrder -> Lexicographic]$

Then the output is as follows:
$b^2d^2h^2g[0] - b^2d^2g[1] - b^2h^2g[1] - d^2h^2g[1] + b^2g[2] + d^2g[2] + h^2g[2] - g[3]$.

Therefore, $g_3 \in \langle g_0, g_1, g_2 \rangle$. Similarly, by using Mathematica, the following holds: $g_4, g_5 \in \langle g_0, g_1, g_2 \rangle$.
Then, it holds the following recurrence formula:

$$g_{k+2} = g_2(b^{2k} + d^{2k} + f^{2k}) - g_1(b^2d^{2k} + b^2f^{2k} + b^{2k}d^{2k} + d^2f^{2k} + b^{2k}f^2 + d^{2k}f^2) + g_0(b^2d^2f^{2k} + b^2d^{2k}f^2 + b^{2k}d^2f^2) + (d^2f^2 + b^2f^2 + b^2d^2)g_k - 2(b^2d^2f^2)g_{k-1}.$$

For $\forall k \geq 2$, it holds as follows: $g_k \in \langle g_0, g_1, g_2 \rangle$. □

Definition 1: Let $G(d,i)$be the sum of all polynomials with that choose $i$chosen from $b_1, \dots, b_d$ and take the product.:

$$G''(1,i) = G''(b_1, b_2, \dots, b_i) := \sum_{j_1=0}^{j_2} \prod_{l=1}^{i} b_{l+j_l}^2,$$

$$G''(k,i) := \sum_{j_k=0}^{j_{k+1}} G''(k-1,i),$$

$$G'(d,1) = G'(b_1, b_2, \dots, b_i) := \sum_{j_1=0}^{d} \prod_{l=1}^{i} b_{l+j_l}^2,$$

$$G'(i,d) := \sum_{j_1=0}^{d} G''(i-1,i),$$

$$G(d,0) := 1, \qquad G'(d,i) := G'(d-i,i)$$

Lemma 3: For polynomials of $2d$-variables

$$F_n(a_1, \dots, a_d, b_1, \dots, b_d) := \sum_{k=1}^{d} a_k b_k^{2n+1},$$

we consider an ideal $I_k = \langle F_0, F_1, F_2, \dots, F_k \rangle$. Then this ideal defines a nondecreasing sequence of ideals.: $\exists k$such that $I_2 = I_k$.

Proof: For the above polynomials of $2d$-variable, we show the following recurrence formula:

$$
\begin{aligned}
F_{k+d-1} &= F_{d-1}\left(b_1^{2k}b_2^{2k}+\cdots+b_d^{2k}\right)\\
&- F_{d-2}\left(b_1^{2}b_2^{2k}+b_1^{2}b_3^{2k}\cdots\right)+F_{d-3}\left(b_1^{2}b_2^{2}b_3^{2k}+\cdots\right)\\
&+\cdots+F_0\left(b_1^{2k}b_2^{2}b_3^{2}\cdots b_d^{2}+\cdots+b_1^{2}b_2^{2}b_3^{2}\cdots b_d^{2k}\right)\\
&+\left(b_1^2b_2^2+b_1^2b_3^2\cdots+b_{d-1}^2b_d^2\right)F_{k+d-3}\\
&-2\left(b_1^2b_2^2b_3^2+\cdots\right)F_{k+d-4}+\cdots\\
&+(-1)^d(d-1)\left(b_1^2b_2^2b_3^2\cdots b_d^2\right)F_{k-1}
\end{aligned}
$$

Let polynomials of 2-variable be $f_l^n=f_l^n(a_l,b_l):=a_lb_l^{2n+1}$, then $F_n(a_1,\dots,a_d,b_1,\dots,b_d)=\sum_{l=1}^d f_l^n$.
Next, we show the following recurrence formula:

$$
\begin{aligned}
\sum_{l=1}^{d} f_l^{k+d-1}
&=\sum_{l=1}^{d}\{f_l^{d-1}\left(b_1^{2k}+b_2^{2k}+\cdots+b_d^{2k}\right)-f_l^{d-2}(b_1^2b_2^{2k}\\
&+b_1^2b_3^{2k}\cdots)+f_l^{d-3}(b_1^2b_2^2b_3^{2k}\\
&+\cdots)+\cdots+f_l^{0}(b_1^{2k}b_2^2b_3^2\cdots b_d^2+\cdots\\
&+b_1^2b_2^2b_3^2\cdots b_d^{2k})+(b_1^2b_2^2+b_1^2b_3^2+\cdots\\
&+b_{d-1}^2b_d^2)f_l^{k+d-3}-2(b_1^2b_2^2b_3^2+\cdots)f_l^{k+d-4}+\cdots\\
&+(-1)^d(d-1)(b_1^2b_2^2b_3^2\cdots b_d^2)f_l^{k-1}\}.
\end{aligned}
$$

For polynomials f_l^n, we show that following recurrence formula holds:

$$
\begin{aligned}
f_l^{k+d-1}
&=f_l^{d-1}\left(b_1^{2k}+b_2^{2k}+\cdots+b_d^{2k}\right)\\
&-f_l^{d-2}(b_1^2b_2^{2k}+b_1^2b_3^{2k}\cdots)+f_l^{d-3}(b_1^2b_2^2b_3^{2k}+\cdots)\\
&+\cdots+f_l^{0}\left(b_1^{2k}b_2^2b_3^2\cdots b_d^2+\cdots+b_1^2b_2^2b_3^2\cdots b_d^{2k}\right)\\
&+(b_1^2b_2^2+b_1^2b_3^2\cdots+b_{d-1}^2b_d^2)f_l^{k+d-3}\\
&-2(b_1^2b_2^2b_3^2+\cdots)f_l^{k+d-4}+\cdots\\
&+(-1)^d(d-1)(b_1^2b_2^2b_3^2\cdots b_d^2)f_l^{k-1}.
\end{aligned}
$$

Let the above first half of the right right-hand side of the above be as follows:

$$
\begin{aligned}
&f_l^{d-1}\left(b_1^{2k}+b_2^{2k}+\cdots+b_d^{2k}\right)-f_l^{d-2}(b_1^2b_2^{2k}+b_1^2b_3^{2k}\cdots)\\
&\quad+f_l^{d-3}(b_1^2b_2^2b_3^{2k}+\cdots)+\cdots\\
&\quad+f_l^{0}\left(b_1^{2k}b_2^2b_3^2\cdots b_d^2+\cdots+b_1^2b_2^2b_3^2\cdots b_d^{2k}\right)\\
&:=\sum_{i=1}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\left(\sum_{j=1}^{d}b_j^{2k}G(d,i-1)|_{b_j=0}\right)\right\}.
\end{aligned}
$$

or $G(d,i-1)$, we define $g_l(d,i-1)$ as follows: $g_l(d,i-1):=G(d,i-1)|_{b_l=0}$.
We divide $\sum_{j=1}^d b_j^{2k}G(d,i-1)|_{b_j=0}$ into terms which contain b_l^{2k} and terms which do not contain b_l^{2k}.

$$
\sum_{j=1}^{d}b_j^{2k}G(d,i-1)|_{b_j=0}=b_l^{2k}g_l(d,i-1)\neq\sum_{j=1,j\neq l}^{d}b_j^{2k}G(d,i-1)|_{b_j=0}.
$$

The above first half of the above satisfies holds as following:

$$
\begin{aligned}
&\sum_{i=1}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\sum_{j=1}^{d}b_j^{2k}G(d,i-1)|_{b_j=0}\right\}\\
&=\sum_{i=1}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\left(b_l^{2k}g_l(d,i-1)+\sum_{j=1,j\neq l}^{d}B\right)\right\}
\end{aligned}
$$

where $B=b_j^{2k}G(d,i-1)|_{b_j=0}$.

Then we prove the following equations hold:

$$
\begin{aligned}
&\sum_{i=1}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\left(b_l^{2k}g(d,i-1)\right)\right\}\\
&\qquad=a_lb_l^{2d-1}b_l^{2k}+\sum_{i=1}^{d-1}(-1)^{i}\times f_l^{d-i-1}b_1^{2k}g(d,i).
\end{aligned}\tag{1}
$$

$$
\sum_{i=1}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\left(\sum_{j=1,j\neq l}^{d}b_j^{2k}G(d,i-1)|_{b_j=0}\right)\right\}=0.\tag{2}
$$

First, we prove equation (1). We divide equation (1) into thefirst term and the other terms:

$$
\begin{aligned}
&\sum_{i=1}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\left(b_l^{2k}g(d,i-1)\right)\right\}\\
&=a_lb_l^{2d-1}b_l^{2k}+\sum_{i=2}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\times\left(b_1^{2k}g(d,i-1)\right)\right\}\\
&=a_lb_l^{2d-1}b_l^{2k}+\sum_{i=2}^{d}(-1)^{i+1}\times f_l^{d-i}b_l^{2k}g(d,i-1).
\end{aligned}
$$

We change i to $i+1$ in all on terms except the first term:

$$
a_lb_l^{2d-1}b_l^{2k}+\sum_{i=1}^{d-1}(-1)^{i}\times f_l^{d-i-1}b_l^{2k}g(d,i).
$$

Next, we prove equation (2).

$$
\begin{aligned}
&\sum_{i=1}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\left(\sum_{j=1,j\neq l}^{d}b_j^{2k}G(d,i-1)|_{b_j=0}\right)\right\}\\
&=\sum_{j=1,j\neq l}^{d}\left\{\sum_{i=1}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\left(b_j^{2k}G(d,i-1)|_{b_j=0}\right)\right\}\right\}.
\end{aligned}
$$

Then we show the following equation holds:

$$
\sum_{i=1}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\left(b_j^{2k}G(d,i-1)|_{b_j=0}\right)\right\}=0.
$$

We divide the equation into the first term and the other terms:

$$
\begin{aligned}
&\sum_{i=1}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\left(b_l^{2k}G(d,i-1)|_{b_j=0}\right)\right\}\\
&\qquad=a_lb_l^{2d-1}b_j^{2k}+\sum_{i=2}^{d}\left\{(-1)^{i+1}\times f_l^{d-i}\times\left(b_j^{2k}G(d,i-1)|_{b_j=0}\right)\right\}\\
&\qquad=a_lb_l^{2d-1}b_j^{2k}+\sum_{i=2}^{d}(-1)^{i+1}\times f_l^{d-i}b_j^{2k}G(d,i-1)\,|_{b_j=0}\\
&\qquad=a_lb_l^{2d-1}b_j^{2k}+\sum_{i=1}^{d-1}(-1)^{i}\times f_l^{d-i-1}b_j^{2k}G(d,i)\,|_{b_j=0}.
\end{aligned}\tag{3}
$$

Then we divide $G(d,i)$ into terms which contain b_l^2 and terms which do not contain b_l^2:
$G(d,i)|_{b_j=0}=b_l^2g_l(d,i-1)|_{b_j=0}+g_l(d,i)|_{b_j=0}$.
Next, we divide the second term into a first term and other terms.

$$
\begin{aligned}
&\sum_{i=1}^{d-1}(-1)^{i}\times f_l^{d-i-1}b_j^{2k}G(d,i)|_{b_j=0}\\
&\qquad=-a_lb_l^{2d-3}b_j^{2k}G(d,1)|_{b_j=0}\\
&\qquad+\sum_{i=2}^{d-1}(-1)^{i}\times f_l^{d-i-1}b_j^{2k}\left(b_l^{2}g_l(d,i-1)|_{b_j=0}+g_l(d,i)|_{b_j=0}\right).
\end{aligned}
$$

We divide $G(d,i)$ into terms which contain b_l^2 and terms which do not contain b_l^2.

$G(d,1)|_{b_j=0} = b_l^2 g_l(d,0)|_{b_j=0} + g_l(d,1)|_{b_j=0}$
$= b_l^2 + g_l(d,1)|_{b_j=0}$,
$-a_l b_l^{2d-3} b_j^{2k}\left(b_l^2 + g_l(d,1)|_{b_j=0}\right) + \sum_{i=2}^{d-1}(-1)^i \times f_l^{d-i-1} b_j^{2k}\left(b_l^2 g_l(d,i-1)|_{b_j=0} + g_l(d,i)|_{b_j=0}\right)$ (4)

By using equations (3) and (4), then equation (2) becomes isas follows:

$$a_l b_l^{2d-1} b_j^{2k} - a_l b_l^{2d-3} b_j^{2k}\left(b_l + g_l(d,1)|_{b_j=0}\right) - \sum_{i=2}^{d-1}(-1)^i \times f_l^{d-i-1} b_j^{2k}\left(b_l^2 g_l(d,i-1)|_{b_j=0} + g_l(d,i)|_{b_j=0}\right).$$

Then, it holds that $f_l^{d-i-1} \times b_l^2 = f_l^{d-i}$, and

$$a_l b_l^{2d-1} b_j^{2k} - a_l b_l^{2d-1} b_j^{2k} - a_l b_l^{2d-3} b_j^{2k} g_l(d,1)|_{b_j=0} + \sum_{i=2}^{d-1}(-1)^i \times f_l^{d-i} b_j^{2k}\, g_l(d,i-1)|_{b_j=0} + \sum_{i=2}^{d-1}(-1)^i \times f_l^{d-i-1} b_j^{2k}\, g_l(d,i)|_{b_j=0}$$
$$= -a_l b_l^{2d-3} b_j^{2k} g(d,1)|_{b_j=0} + \sum_{i=2}^{d-1}(-1)^i \times f_l^{d-i} b_j^{2k}\, g_l(d,i-1)|_{b_j=0} + \sum_{i=2}^{d-1}(-1)^i \times f_l^{d-i-1} b_j^{2k}\, g_l(d,i)|_{b_j=0}.$$

We change i to $i-1$ in $\sum_{i=2}^{d-1}(-1)^i \times f_l^{d-i-1} b_j^{2k} g_l(d,i)|_{b_j=0}$,

$$-a_l b_l^{2d-3} b_j^{2k} g_{l(d,1)|_{b_j=0}} + \sum_{i=2}^{d-1}(-1)^i \times f_l^{d-i} b_j^{2k} g_l(d,i-1)|_{b_j=0} + \sum_{i=2}^{d}(-1)^{i+1} \times f_l^{d-i} b_j^{2k} g_l(d,i-1)|_{b_j=0}.$$

Then all terms of the above polynomial except the first term of above polynomial are equal to equals the second term and the last term of the above polynomial, and it holds that $g_l(d,d-1)|_{b_j=0} = 0$ and

$$-a_l b_l^{2d-3} b_j^{2k} g_l(d,1)|_{b_j=0} + (-1)^2 \times f_l^{d-2} b_j^{2k} g_l(d,1)|_{b_j=0} + (-1)^{d+1} \times f_l^0 b_j^{2k} g(d,d-1)|_{b_j=0} = 0.$$

Let the above second half of right right-hand side of the above be as follows:

$$(b_1^2 b_2^2 + b_1^2 b_3^2 \cdots + b_{d-1}^2 b_d^2) f_l^{k+d-3} - 2(b_1^2 b_2^2 b_3^2 + \cdots) f_l^{k+d-4} + \cdots + (-1)^d (d-1)(b_1^2 b_2^2 b_3^2 \cdots b_d^2) f_1^{k-1} \coloneqq \sum_{i=1}^{d-1}(-1)^{i+1} i \times f_l^{k+d-i-2} G(d,i+1)$$

We show that the following equation holds:

$$\sum_{i=1}^{d-1}(-1)^{i+1} i \times f_l^{k+d-i-2} G(d,i+1) = \sum_{i=1}^{d-1}(-1)^{i+1} \times f_l^{k+d-i-2} b_l^2 g_l(d,i). \quad (5)$$

We divide equation (5) into the last term and the other terms:

$$\sum_{i=1}^{d-2}(-1)^{i+1} i \times f_l^{k+d-i-2} G(d,i+1) + (-1)^d (d-1) f_l^{k-1} G(d,d).$$

We divide $G(d,i+1)$ into terms which contain b_l^2 and termswhich do not contain b_l^2:

$G(d,i+1) = b_l^2 g_l(d,i) + g_l(d,i+1)$,
$G(d,d) = b_l^2 g_l(d,d-1) + g_l(d,d) = b_l^2 g_l(d,d-1)$,

$$\sum_{i=1}^{d-2}(-1)^{i+1} i \times f_l^{k+d-i-2}\left(b_l^2 g_l(d,i) + g_l(d,i+1)\right) + (-1)^d (d-1) f_l^{k-1} b_l^2 g(d,d-1)$$
$$= \sum_{i=1}^{d-2}(-1)^{i+1} i \times f_l^{k+d-i-2} b_l^2 g_l(d,i) + \sum_{i=1}^{d-2}(-1)^{i+1} i \times f_l^{k+d-i-2} g_l(d,i+1) + (-1)^d (d-1) f_l^{k-1} b_l^2 g_l(d,d-1).$$

The terms except the last term are as follows:

$$\sum_{i=1}^{d-1}(-1)^{i+1} i \times f_l^{k+d-i-2} b_l^2 g_l(d,i) + \sum_{i=1}^{d-2}(-1)^{i+1} i \times f_l^{k+d-i-2} g_l(d,i+1).$$

We divide $\sum_{i=1}^{d-1}(-1)^{i+1} i \times f_l^{k+d-i-2} b_l^2 g_l(d,i)$ into the firstterm and the other terms and we change i to $i-1$ in $\sum_{i=1}^{d-2}(-1)^{i+1} i \times f_l^{k+d-i-2} g_l(d,i+1)$:

$$(-1)^2 f_l^{k+d-3} b_l^2 g_l(d,1) + \sum_{i=2}^{d-1}(-1)^{i+1} i f_l^{k+d-i-2} b_l^2 g_l(d,i) \quad (6)$$

$$+\sum_{i=2}^{d-1}(-1)^i (i-1) f_l^{k+d-i-1} g_l(d,i). \quad (7)$$

By using the relation $i = 1 + (i-1)$, equation (6) becomes is as follows:

$$(-1)^2 \times f_l^{k+d-3} b_l^2 g_l(d,1) + \sum_{i=2}^{d-1}(-1)^{i+1} i \times f_l^{k+d-i-2} b_l^2 g_l(d,i)$$
$$= (-1)^2 \times f_l^{k+d-3} b_l^2 g_l(d,1) + \sum_{i=2}^{d-1}(-1)^{i+1}\{1 + (i-1)\} \times f_l^{k+d-i-2} b_l^2 g_l(d,i)$$
$$= (-1)^2 \times f_l^{k+d-3} b_l^2 g_l(d,1) + \sum_{i=2}^{d-1}(-1)^{i+1} \times f_l^{k+d-i-2} b_l^2 g_l(d,i) + \sum_{i=2}^{d-1}(-1)^{i+1}(i-1) \times f_l^{k+d-i-2} b_l^2 g_l(d,i).$$

Then the following holds as follows:

$$\sum_{i=1}^{d-1}(-1)^{i+1} \times f_l^{k+d-i-2} b_l^2 g_l(d,i) + \sum_{i=2}^{d-1}(-1)^{i-1}(i-1) \times f_l^{k+d-i-2} b_l^2 g_l(d,i) \quad (8)$$

By using equations (6), (7) and (8), then equation (5) becomes as follows:

$$\sum_{i=1}^{d-1}(-1)^{i+1}(i-1)\times f_l^{k+d-i-1}g_l(d,i)$$
$$+\sum_{i=1}^{d-1}(-1)^{i+1}\times f_l^{k+d-i-2}b_l^2g_l(d,i)$$
$$+\sum_{i=2}^{d-1}(-1)^{i-1}(i-1)\times f_l^{k+d-i-2}b_l^2g_l(d,i)\,.$$

Then the following holds:

$$\sum_{i=2}^{d-1}(-1)^{i}(i-1)\times f_l^{k+d-i-1}b_l^2g_l(d,i)$$
$$+\sum_{i=2}^{d-1}(-1)^{i-1}(i-1)\times f_l^{k+d-i-1}g_l(d,i)=0.$$

Thus, the following recurrence formula holds:

$$\mathrm{a_l b_l^{2d-1} b_l^{2k}}+\sum_{\mathrm{i=1}}^{\mathrm{d-1}}(-1)^{\mathrm{i}}\times \mathrm{f_l^{d-i-1}b_1^{2k}g(d,i)}$$
$$+\sum_{\mathrm{i=1}}^{\mathrm{d-1}}(-1)^{\mathrm{i+1}}\times \mathrm{f_l^{k+d-i-2}b_l^2g_l(d,i)}\,.$$

The following also holds:

$$\sum_{\mathrm{i=1}}^{\mathrm{d-1}}(-1)^{\mathrm{i}}\times \mathrm{f_l^{d-i-1}b_1^{2k}g(d,i)}$$
$$+\sum_{\mathrm{i=1}}^{\mathrm{d-1}}(-1)^{\mathrm{i+1}}\times \mathrm{f_l^{k+d-i-1}g_l(d,i)}=0.$$

Therefore, for $k\geq d-1$, $F_k\in\langle F_0,F_1,F_2,\dots,F_{d-1}\rangle$. □

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