ISSN:2320-0790

COMPUSOFT
An International Journal of Advanced Computer Technology

Error Estimations in an Approximation on a Compact Interval with a Wavelet Bases

Dr. Marco Schuchmann[1], Michael Rasguljajew[2]

[1,2]Hochschule Darmstadt

Abstract: By an approximation with a wavelet base we have in practise not only an error if the function y is not in V_j. There we have a second error because we do not use all bases functions. If the wavelet has a compact support we have no error by using only a part of all basis function. If we need an approximation on a compact interval I (which we can do even if y is not quadratic integrable on *R*, because in that case it must only be quadratic integrable on I) leads to worse approximations if we calculate an orthogonal projection from $1_I y$ in V_j. We can get much better approximations, if we apply a least square approximation with points in I. Here we will see, that this approximation can be much better than a orthogonal projection form y or $1_I y$ in V_j. With the Shannon wavelet, which has no compact support, we saw in many simulations, that a least square approximation can lead to much better results than with well known wavelets with compact support. So in that article we do an error estimation for the Shannon wavelet, if we use not all bases coefficients.

Keywords: component; formatting; style; styling; (insert minimum 5 to 8 key words)

I. INTRODUCTION

In the wavelet theory a scaling function ϕ is used, which has properties that are defined in the MSA (multi scale analysis). Through the MSA we know, that we can construct an orthonormal basis of a closed subspace V_j, where V_j belongs to a sequence of subspaces with the following property:

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(R),$$

$\{\phi_{j,k}(t)\}_{k\in Z}$ is an orthonormal basis of V_j with $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$.

We use the following approximation function:

$$y_j(t) := \sum_{k=k_{\min}}^{k_{\max}} c_k \cdot \phi_{j,k}(t)$$

k_{max} and k_{min} depend on the approximation interval $I = [t_0, t_{end}]$.

For easier notation we assume $k_{max} = -k_{min}$.

For $k_{max} < \infty$ we get the following error estimation:

$$\left\| \sum_k c_k \phi_{j,k} - \sum_{|k|\le k_{\max}} c_k \phi_{j,k} \right\| = \sqrt{\sum_{|k|>k_{\max}} |c_k|^2}$$

‖.‖ is the $L^2(R)$ norm.

Here we get the following basis coefficients:

(1a) $$c_k = \langle y, \phi_{j,k} \rangle = \int_R y(t) \cdot \phi_{j,k}(t) dt$$

In the case, that y is not quadratic integrable on R and we need an approximation on the Interval *I*, we can calculate an orthogonal projection of $1_I\, y$ on V_j and we get the following basis coefficients:

(1b) $$c_k = \langle 1_I\, y, \phi_{j,k} \rangle = \int_I y(t) \cdot \phi_{j,k}(t) dt$$

That leads in general to a worse approximation, if j is not very big (see [7]).

Now we use the Shannon wavelet and assume for $I = [-a, a]$, $a > 0$ and $k_0 := k_{max} - \lceil 2^j a \rceil$ where $\lceil x \rceil$ is the smallest integer *n* with $n \ge x$. $c_k = \langle 1_I\, y, \phi_{j,k} \rangle$. Additional we assume, that $k_{max} > [2^j a]$. Then we get the following error estimation:

(2)

$$\left\| \sum_k c_k \phi_{j,k} - \sum_{|k|\le k_{\max}} c_k \phi_{j,k} \right\| = \sqrt{\sum_{|k|>k_{\max}} |c_k|^2} \le \frac{2^{j/2} c}{\pi} \sqrt{\xi(2) - \sum_{k=1}^{k_0} \frac{1}{k^2}}$$

Here ζ is the Riemann ζ function and c depends on *a* and *y*.

Proof:

$$|c_k| = \left| \int_I y(t) \cdot \phi_{j,k}(t) dt \right| \le \int_I |y(t) \cdot \phi_{j,k}(t)| dt$$

$$\le 2a \cdot \max_{t\in I} |y(t) \cdot \phi_{j,k}(t)| \le \underbrace{2a \cdot \max_{t\in I} |y(t)|}_{:=c} \cdot \max_{t\in I} |\phi_{j,k}(t)|$$

Now we consider the last factor:

$$\max_{t\in I} |\phi_{j,k}(t)| = 2^{j/2} \cdot \max_{t\in I} \left| \frac{\sin(\pi(2^j t - k))}{\pi(2^j t - k)} \right| \le \frac{2^{j/2}}{\pi} \max_{t\in I} \left| \frac{1}{2^j t - k} \right|$$

Because of $k > 2^j \cdot a$

$$\left| \frac{1}{2^j t - k} \right|$$

has on I = [-a, a] it's maximum at the point t = a.

So we get

$$\max_{t\in I} |\phi_{j,k}(t)| \le \frac{2^{j/2}}{\pi} \frac{1}{k - 2^j a}$$

and

$$|c_k| \le \frac{2^{j/2} c}{\pi} \frac{1}{k - 2^j a} \quad .$$

Now we get:

$$\left\| \sum_k c_k \phi_{j,k} - \sum_{|k|\le k_{\max}} c_k \phi_{j,k} \right\| \le \frac{2^{j/2} c}{\pi} \sqrt{\sum_{k=k_{\max}+1}^{\infty} \frac{1}{(k - 2^j a)^2}}$$

If we set $k_0 = k_{\max} - 2^j a$ if *a* is an integer or generally $k_0 = k_{\max} - [2^j a]$, then we get the proposition:

$$\left\| \sum_k c_k \phi_{j,k} - \sum_{|k|\le k_{\max}} c_k \phi_{j,k} \right\| \le \frac{2^{j/2} c}{\pi} \sqrt{\sum_{k=k_0+1}^{\infty} \frac{1}{k^2}}$$

In the following example we will see, that if we calculate the coefficients c_k by the minimization of

(3) $$Q(c) = \sum_{i=0}^{m} \left(y_j(t_i) - y(t_i) \right)^2$$

we can get much better results as if we calculate the coefficients over (1a) or (1b). In the example we use equidistant points t_i (that means $t_{i+1} - t_i = \Delta t$). The method we call the (discrete) least square method.

Generally we know that

$$\left\| y - y_j^R \right\|_{L^2(I)} \ge \left\| y - y^I{}_j \right\|_{L^2(I)}$$

if $\min_{g\in V_j} \| y - g \| = \| y - y_j^R \|$

and

$$\min_{g\in V_j} \| y - g \|_{L^2(I)} = \| y - y_j^I \|_{L^2(I)} .$$

So here y_j^R is the best approximation (in reference to the L^2 norm) of *y* on *R* and y_j^I is it's best approximation on *I*.

II. EXAMPLES

In all examples we set $j = 1$. The approximation interval *I* is [-1, 1]. We now show three examples, one with a $L^2(R)$ function, one with a function in V_1 and in the last example, we approximate a function, which is not in $L^2(R)$ but in $L^2(I)$. Here we project $1_{[-1,1]}y$ in V_1.

In all examples we compare the orthogonal projection from y with the least square estimation (see (3)) with $t_0 = -1$.

1) *y* is in $L^2(R)$:

We use the function $y(t) = e^{-t^2}$ and set $k_{max} = 15$.

a) Orthogonal projection on V_1:

Here is the graph of *y* (dashed) and y_j:

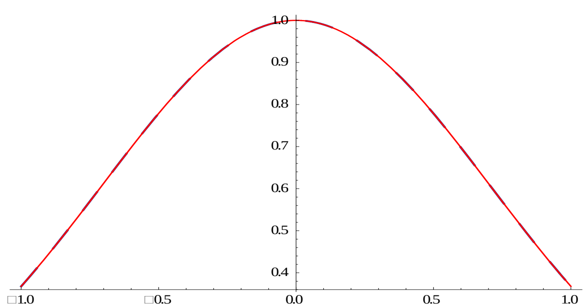


Figure 1.

The graph of $y - y_j$:

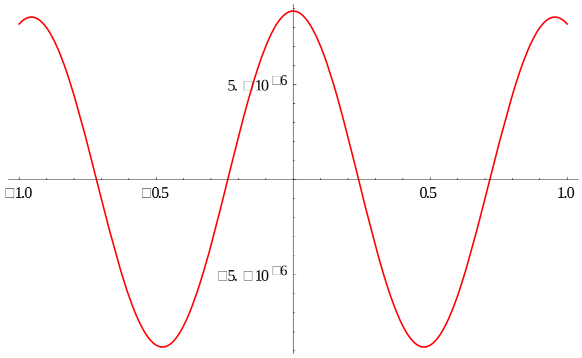


Figure 2.

For y and y_j on [-3,3]*:*

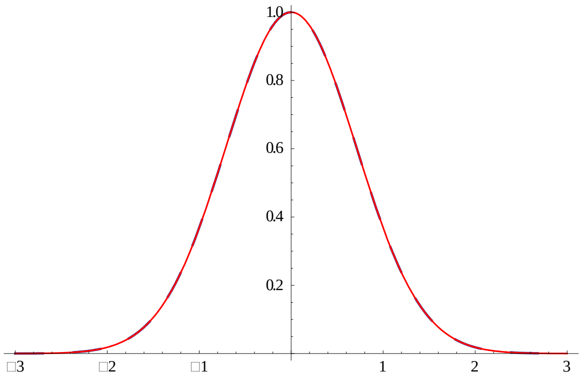


Figure 3.

For $y - y_j$ on [-3,3]*:*

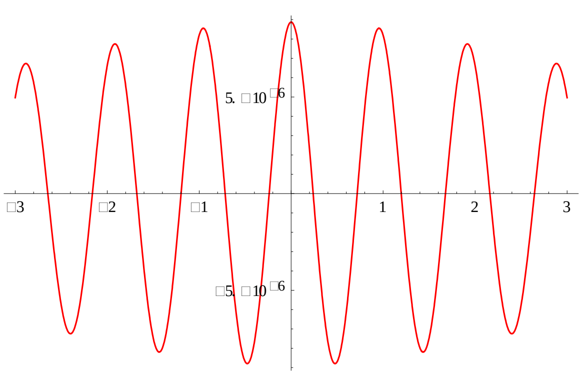


Figure 4.

Now we see the approximation function calculated through the least square method (3).

b) Least square approximation like in (3) described. We set Δt = 1/20.

Here is the graph of y (dashed) and y_j*:*

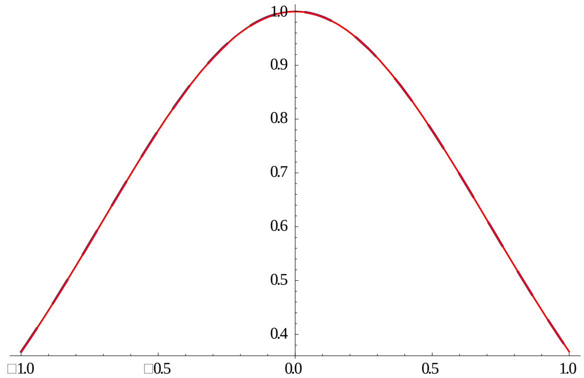


Figure 5.

The graph of $y - y_j$:

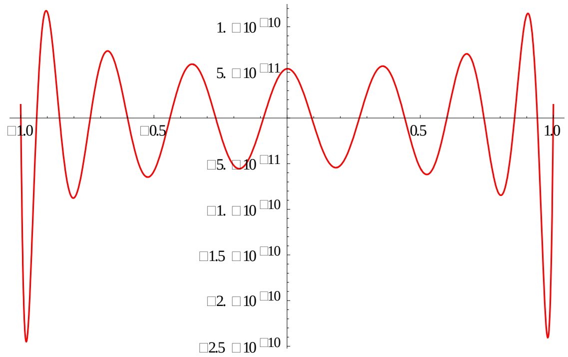


Figure 6.

Here we see that the approximation on the interval [-1, 1] can even be used for an extrapolation:

For y and y_j on [-3,3] (extrapolation)*:*

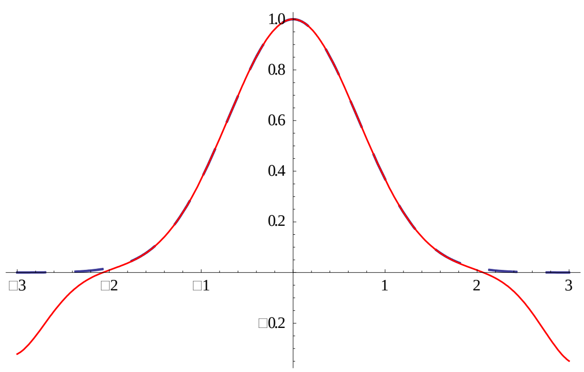


Figure 7.

For $y - y_j$ on [-3,3] (extrapolation)*:*

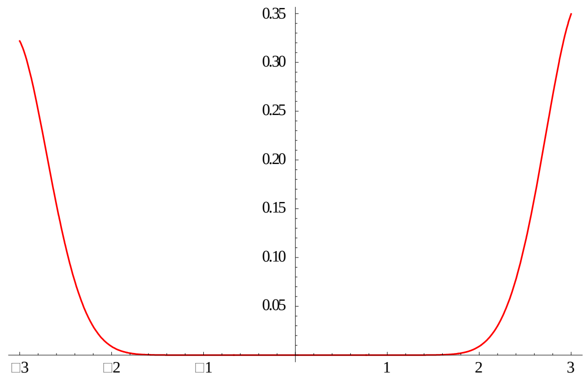


Figure 8.

Here we see a table of the coefficients calculated like in (1a) named $c_k^{L^2(R)}$ and the coefficients calculated over the least square approach named c_k.

k	$c_k^{L^2(R)}$	c_k	$\lvert(c_k^{L^2(R)} - c_k)/ c_k^{L^2(R)}\rvert$
□15	9.98646 □10□7	0.0549149	54988.4
□14	□1.12346 □10□6	□0.162391	144544.
□13	1.27115 □10□6	0.156844	123386.
□12	□1.44692 □10□6	0.0958456	66242.3
□11	1.65714 □10□6	□0.11707	70647.
□10	□1.90945 □10□6	□0.143967	75396.
□9	2.2138 □10□6	□0.00448466	2026.77
□8	□2.49637 □10□6	0.309462	123966.
□7	6.39081 □10□6	□0.0238892	3739.06
□6	0.0000837542	□0.227517	2717.49
□5	0.00136911	□0.0770245	57.2587
□4	0.0129464	0.00674154	0.479275
□3	0.0745338	0.0744649	0.000924883
□2	0.260124	0.26013	0.0000222773
□1	0.550701	0.550695	0.0000111669
0	0.707101	0.707107	8.87617 □10□6
1	0.550701	0.550695	0.0000111669
2	0.260124	0.26013	0.0000222773
3	0.0745338	0.0744648	0.000926093
4	0.0129464	0.00663831	0.487248
5	0.00136911	□0.0799417	59.3894
6	0.0000837542	□0.247064	2950.86
7	6.39081 □10□6	□0.0756419	11837.
8	□2.49637 □10□6	0.193832	77646.3
9	2.2138 □10□6	□0.254212	114832.
10	□1.90945 □10□6	□0.282422	147907.
11	1.65714 □10□6	0.125028	75447.2
12	□1.44692 □10□6	0.124006	85704.4
13	1.27115 □10□6	□0.0955985	75207.3
14	□1.12346 □10□6	0.0000702266	63.5093
15	9.98646 □10□7	0.287446	287835.

We get similar results with a smaller Δt = 1/100:

k	$c_k^{L^2(R)}$	c_k	$\lvert(c_k^{L^2(R)} - c_k)/ c_k^{L^2(R)}\rvert$
□15	9.98646 □10□7	0.0555138	55588.1
□14	□1.12346 □10□6	0.00279287	2486.95
□13	1.27115 □10□6	0.0908575	71475.6
□12	□1.44692 □10□6	□0.0753896	52102.6
□11	1.65714 □10□6	□0.182931	110391.
□10	□1.90945 □10□6	□0.274183	143592.
□9	2.2138 □10□6	0.0337608	15249.2
□8	□2.49637 □10□6	0.354246	141905.
□7	6.39081 □10□6	□0.0793108	12411.1
□6	0.0000837542	□0.250574	2992.78
□5	0.00136911	□0.0757508	56.3283
□4	0.0129464	0.00727581	0.438006
□3	0.0745338	0.0744744	0.000797106
□2	0.260124	0.26013	0.0000222771
□1	0.550701	0.550695	0.0000111669
0	0.707101	0.707107	8.87617 □10□6
1	0.550701	0.550695	0.0000111669
2	0.260124	0.26013	0.000022277
3	0.0745338	0.0744823	0.000690303
4	0.0129464	0.0086523	0.331684
5	0.00136911	□0.0482002	36.2054
6	0.0000837542	□0.115958	1385.51
7	6.39081 □10□6	0.0431607	6752.56
8	□2.49637 □10□6	0.0832123	33334.3
9	2.2138 □10□6	□0.226748	102426.
10	□1.90945 □10□6	□0.00943628	4940.88
11	1.65714 □10□6	0.154941	93498.1
12	□1.44692 □10□6	0.186859	129144.
13	1.27115 □10□6	0.0550771	43327.5
14	□1.12346 □10□6	□0.132146	117623.
15	9.98646 □10□7	0.138983	139171.

Here we see the errors $\left\| y - y_j \right\|_{L^2([-1,1])}$ for the different approximation methods:

$c_k^{L^2(R)}$:

8.92392×10^{-6}

least square c_k with Δt=1/20:

9.00771×10^{-11}

least square c_k with Δt=1/100:

6.4122×10^{-11}

And here the errors for a bigger interval (extrapolation) $\left\| y - y_j \right\|_{L^2([-1.5,1.5])}$:

$c_k^{L^2(R)}$:

0.0000107337

least square c_k with Δt=1/20:

0.0000248848

least square c_k with Δt=1/100:

0.0000196081

2) *y* is in *V*$_1$:

Here we use a part of the orthonormal basis of V1 to construct a function in V1:

$$y(t) = \sum_{k=-25}^{25} c_k \cdot \phi_{j,k}(t) \quad \text{with} \quad c_k=(-1)^k/(k^2+1).$$

a) Orthogonal projection on V_1:

We set k_{max} = *15* (so $y_j \neq$ y).

Here is the graph of *y* (dashed) and y_j:

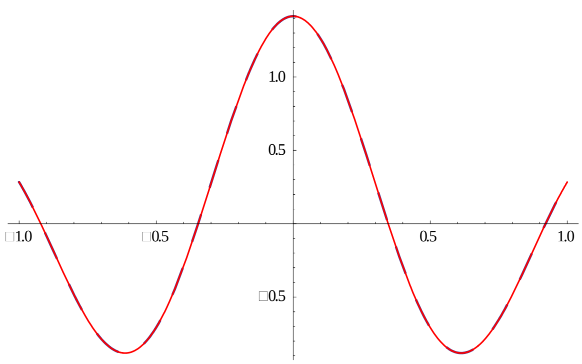


Figure 9.

The graph of $y - y_j$:

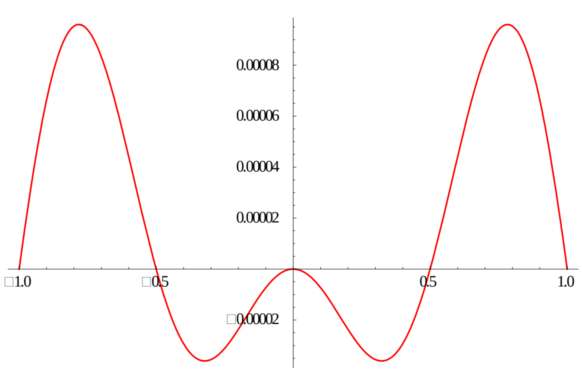


Figure 10.

y(t) and $y_j(t)$ on [-3,3]:

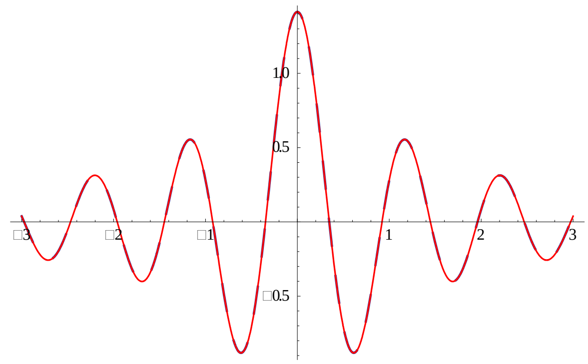


Figure 11.

For $y - y_j$ on [-3,3]:

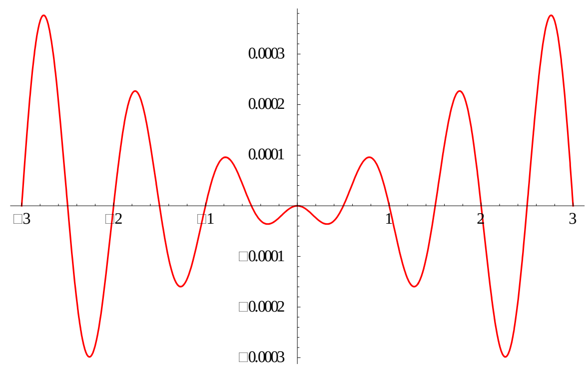


Figure 12.

b) Least square approximation like in (3) described k_{max} = 10 and Δt = 1/20:

Here is the graph of *y* (dashed) and y_j:

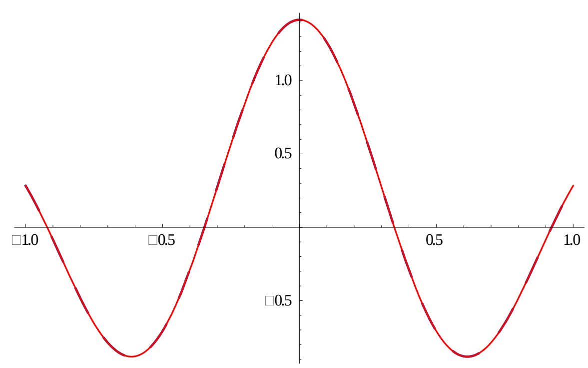


Figure 13.

The graph of $y - y_j$:

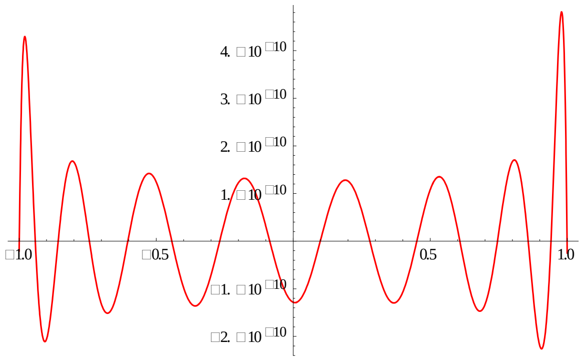


Figure 14.

For y and y_j on [-3,3] (extrapolation):

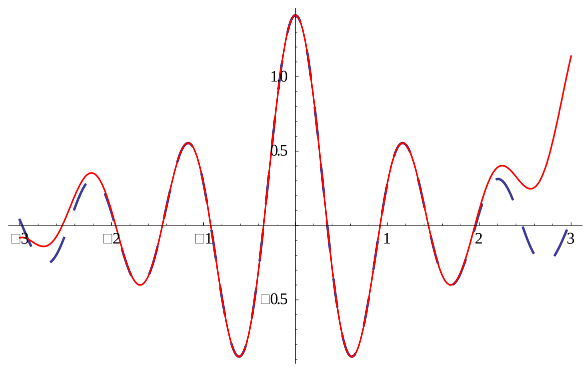


Figure 15.

For y - y_j on [-3,3] (extrapolation):

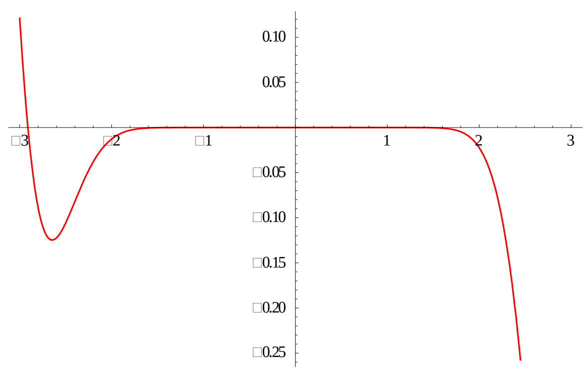


Figure 16.

□10.	0.00990099	4.16328	419.491
□9.	□0.0121951	5.46063	448.772
□8.	0.0153846	0.84032	53.6208
□7.	□0.02	□0.911112	44.5556
□6.	0.027027	□0.0585332	3.16573
□5.	□0.0384615	0.0363541	1.94521
□4.	0.0588235	0.0678428	0.153328
□3.	□0.1	□0.0998917	0.00108337
□2.	0.2	0.2	7.19026 □10□11
□1.	□0.5	□0.5	1.75861 □10□10
0.	1.	1.	9.0701 □10□11
1.	□0.5	□0.5	1.52477 □10□10
2.	0.2	0.2	8.76749 □10□11
3.	□0.1	□0.0998525	0.00147549
4.	0.0588235	0.0746509	0.269064
5.	□0.0384615	0.184991	5.80977
6.	0.027027	0.805369	28.7987
7.	□0.02	0.396838	20.8419
8.	0.0153846	□0.6829	45.3885
9.	□0.0121951	0.626755	52.3939
10.	0.00990099	1.40537	140.943

For a comparison we set $k_{max} = 15$ and $\Delta t = 1/100$:

k	$c_k^{L^2(R)}$	c_k	$\lvert(c_k^{L^2(R)} - c_k)/ c_k^{L^2(R)}\rvert$
□15.	□0.00442478	0.171272	39.7075
□14.	0.00507614	□0.0487982	10.6132
□13.	□0.00588235	0.497604	85.5926
□12.	0.00689655	0.247044	34.8214
□11.	□0.00819672	0.117465	15.3307
□10.	0.00990099	0.621908	61.8127
□9.	□0.0121951	□0.0267993	1.19754
□8.	0.0153846	□0.196127	13.7482
□7.	□0.02	□0.205125	9.25624
□6.	0.027027	□0.177672	7.57388
□5.	□0.0384615	□0.1073	1.78981
□4.	0.0588235	0.0531263	0.096853
□3.	□0.1	□0.10006	0.000604441
□2.	0.2	0.2	6.2635 □10□10
□1.	□0.5	□0.5	8.38828 □10□11
0.	1.	1.	4.66002 □10□11
1.	□0.5	□0.5	8.86247 □10□11
2.	0.2	0.2	6.92697 □10□10
3.	□0.1	□0.100071	0.000711496
4.	0.0588235	0.0515767	0.123196
5.	□0.0384615	□0.135855	2.53223
6.	0.027027	□0.308153	12.4017
7.	□0.02	□0.318697	14.9348
8.	0.0153846	□0.0523028	4.39968
9.	□0.0121951	□0.204147	15.74
10.	0.00990099	□0.0272149	3.74871
11.	□0.00819672	□0.137145	15.7316
12.	0.00689655	0.0496725	6.20252
13.	□0.00588235	0.218437	38.1342
14.	0.00507614	□0.209121	42.1968
15.	□0.00442478	0.0107344	3.42598

k	$c_k^{L^2(R)}$	c_k	$\lvert(c_k^{L^2(R)} - c_k)/ c_k^{L^2(R)}\rvert$

Here we see the errors $\left\| y - y_j \right\|_{L^2([-1,1])}$ for the different approximation methods:

$c_k^{L^2(R)}$:

0.0000708509

least square c_k with Δt=1/20 k_{max} = 10:

1.80875×10^{-10}

least square c_k with Δt=1/100 k_{max} = 15:

8.20892×10^{-11}

And here the errors for a bigger interval (extrapolation) $\left\| y - y_j \right\|_{L^2([-1.5,1.5])}$:

$c_k^{L^2(R)}$:

0.00013277

least square c_k with Δt=1/20 k_{max} = 10:

0.000050488

least square c_k with Δt=1/100 k_{max} = 15:

0.0000257716

For k_{max} = 15 and Δt = 1/100:

The graph of y - y_j :

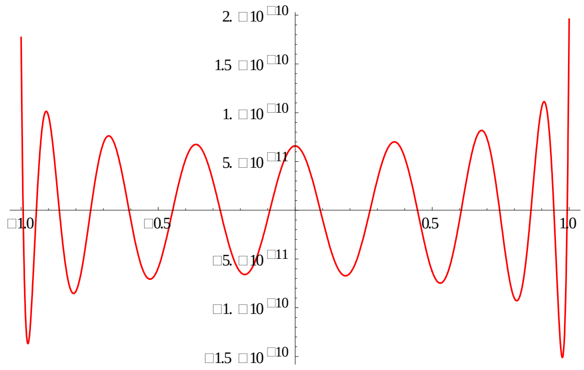


Figure 17.

For y - y_j on [-3,3] (extrapolation):

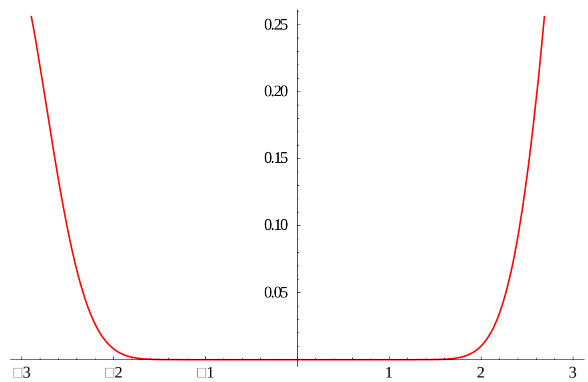


Figure 18.

3) y is in Λ^2([-1,1]):

Here we use the non $L^2(R)$ function $y(t) = e^{-t}$.

a) At first we calculate the orthogonal projection of $1_{[-2,2]}y$ on V_1 (like in (1b) described):

Here is the graph of y (dashed) and y_j:

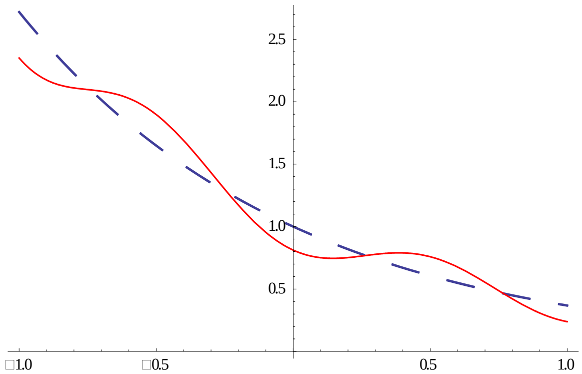


Figure 19.

The graph of $y - y_j$:

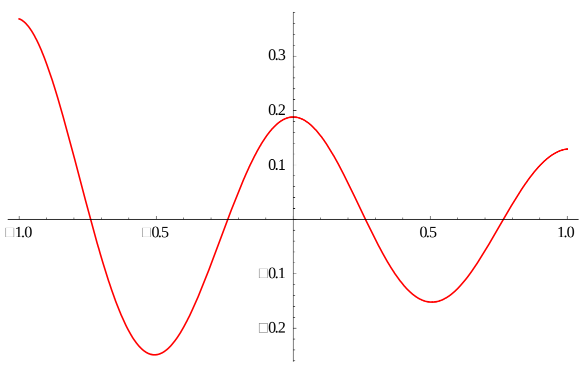


Figure 20.

b) Least square approximation like in (3) described with k_{max} = 15 and Δt = 1/20:

Here is the graph of y (dashed) and y_j:

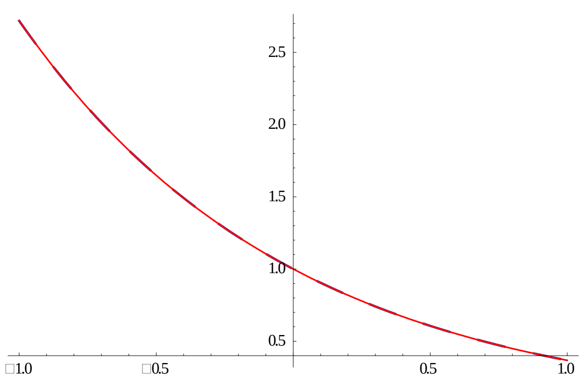


Figure 21.

The graph of $y - y_j$:

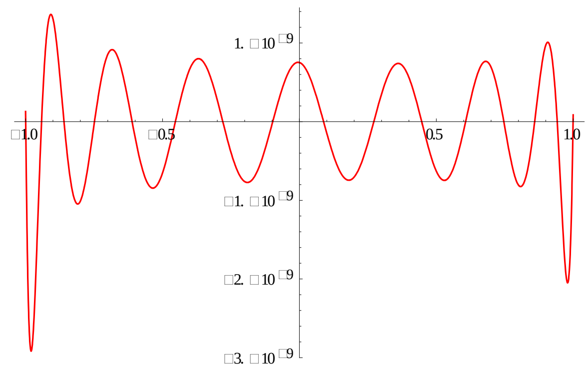


Figure 22.

For y and y_j on [-3,3] (extrapolation):

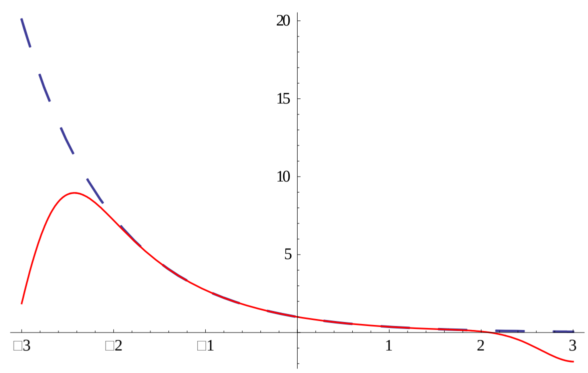


Figure 23.

For $y - y_j$ on [-3,3] (extrapolation):

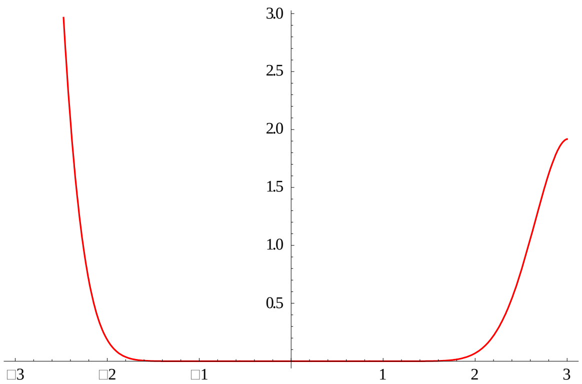


Figure 24.

k	$c_k^{L^2(R)}$	c_k	$\lvert(c_k^{L^2(R)} - c_k)/ c_k^{L^2(R)}\rvert$
□15	□0.045956	0.636832	14.8574
□14	0.0505131	3.9634	77.4629
□13	□0.0560689	0.40329	8.19276
□12	0.062991	□1.79733	29.5331
□11	□0.0718511	0.709774	10.8784
□10	0.0835905	□1.46529	18.5293
□9	□0.0998736	2.04353	21.4612
□8	0.123941	□1.32979	11.7292
□7	□0.163037	□5.97515	35.649
□6	0.237215	1.31779	4.55524
□5	□0.428941	6.26748	15.6115
□4	2.34877	5.09408	1.16883
□3	3.6603	3.16799	0.134501
□2	1.6616	1.92212	0.156783
□1	1.34176	1.16582	0.131127
0	0.574112	0.707107	0.231653
1	0.536393	0.428882	0.200433
2	0.168756	0.26013	0.541458
3	0.240166	0.157235	0.345307
4	□0.0124894	0.0470357	4.76604
5	0.0488908	□0.492759	11.0788
6	□0.0473013	□1.32103	26.928
7	0.0440978	□0.00904417	1.20509
8	□0.0409305	0.563427	14.7655
9	0.0380603	□0.99555	27.1572
10	□0.0355111	1.25702	36.3981
11	0.0332544	0.215155	5.46998
12	□0.0312522	□0.117797	2.76924
13	0.0294685	0.369592	11.5419
14	□0.0278719	□0.961601	33.5008
15	0.0264356	1.18362	43.7738

Here we see the errors $\|y - y_j\|_{L^2([-1,1])}$ for the different approximation methods:

$c_k^{L^2(R)}$:

0.219725

least square c_k with Δt=1/20:

1.01568×10^{-9}

least square c_k with Δt=1/100:

4.90045×10^{-10}

And here the errors for a bigger interval (extrapolation) $\|y - y_j\|_{L^2([-1.5,1.5])}$:

$c_k^{L^2(R)}$:

0.353685

least square c_k with Δt=1/20:

0.00032093

least square c_k with Δt=1/100:

0.000178703

In all examples the least square method (which is a numerical approach to get the "best" solution on the approximation interval I) leads to very good results. Even to better results as with the best L^2 approximation on R.

III. Conclusion

In all examples the least square method (which is a numerical approach to get the "best" solution on the approximation interval I) leads to very good results. Even to better results as with the best L^2 approximation on R.

A. *Text Font of Entire Document*

The entire document should be in Times New Roman or Times font. Other font types may be used if needed for special purposes. Type 3 fonts should not be used. Recommended font sizes are shown in Table 1.

B. *Title and Author Details*

Title must be in 20 points Times New Roman font. Author name must be in 11 points times new roman font. Author affiliation must be in 10 points italic Times new roman. Email address must be in 10 points times new roman font.

All title and author details must be in single-column format and must be centered. Every word in a title must be capitalized. Email address is compulsory for the corresponding author.

C. Section Headings

No more than three levels of headings should be used. All headings must be in 10pt font. Every word in a heading must be capitalized except for short minor words as listed in Section III-B.

Level-1 Heading: A level-1 heading must be in Small Caps, centered and numbered using uppercase Roman numerals. For example, see heading "III Page Style" of this document. The two level-1 headings which must not be numbered are "Acknowledgment" and "References".

Level-2 Heading: A level-2 heading must be in Italic, left-justified and numbered using an uppercase alphabetic letter followed by a period. For example, see heading "C. Section Headings" above.

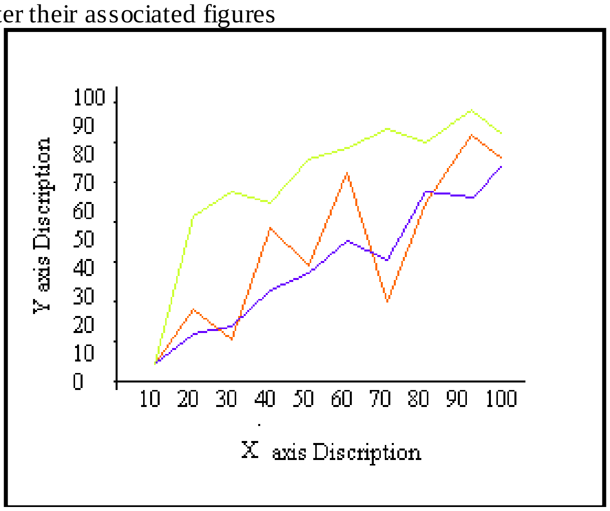
Level-3 Heading: A level-3 heading must be indented, in Italic and numbered with an Arabic numeral followed by a right parenthesis. The level-3 heading must end with a colon. The body of the level-3 section immediately follows the level-3 heading in the same paragraph. For example, this paragraph begins with a level-3 heading.

D. Figures and Tables

Figures and tables must be centered in the column. Large figures and tables may span across both columns. Any table or figure that takes up more than 1 column width must be positioned either at the top or at the bottom of the page.

E. Figure Captions

Figures must be numbered using Arabic numerals. Figure captions must be in 8 pt Regular font. Captions of a single line must be centered whereas multi-line captions must be justified. Captions with figure numbers must be placed after their associated figures



F. Table Captions

Tables must be numbered using uppercase Roman numerals. Table captions must be centred and in 8 pt Regular font with Small Caps. Every word in a table caption must be capitalized except for short minor words as listed in Section III-B. Captions with table numbers must be placed before their associated tables, as shown in Table

Sr. No .	***Heading 1***	***Heading 2***	***Heading 3***	***Headin g 4***	***Heading 5***	***Headin g 6***

G. Page Numbers, Headers and Footers

Page numbers, headers and footers must not be used.

H. Links and Bookmarks

All hypertext links and section bookmarks will be removed from papers during the processing of papers for publication. If you need to refer to an Internet email address or URL in your paper, you must type out the address or URL fully in Regular font.

IV. References

The heading of the References section must not be numbered. All reference items must be in 8 pt font. Please use Regular and Italic styles to distinguish different fields as shown in the References section. Number the reference items consecutively in square brackets (e.g. [1]).

[1] M. Schuchmann, M. Rasguljajew, 2013. "Error Estimation of an Approximation in a Wavelet Collocation Method". Journal of Applied Computer Science & Mathematics, No. 14 (7) / 2013, Suceava. (http://jacs.usv.ro/index.php?pag=showcontent&issue=14&year=2013).

[2] M. Schuchmann, M. Rasguljajew, 2013. "Error Estimation and Assessment of an Approximation in a Wavelet Collocation Method". American Journal of Computational Mathematics, Vol.3, No.2, June 2013.

[3] M. Schuchmann, M. Rasguljajew, 2013. "An Approximation on a Compact Interval Calculated with a Wavelet Collocation Method can Lead to Much Better Results than other Methods". Journal of Approximation Theory and Applied Mathematics, Vol. 1.

[4] M. Schuchmann, M. Rasguljajew, 2013. "Extrapolation and Approximation with a Wavelet Collocation Method for ODEs". Journal of Approximation Theory and Applied Mathematics, Vol. 1.

[5] M. Schuchmann, M. Rasguljajew, 2013. "Determination of Optimal Parameters in a Wavelet Collocation Method". International Journal of Emerging Technology and Advanced Engineering, Vol. 3, Issue 5, May 2013. http://www.ijetae.com/files/Volume3Issue5/IJETAE_0513_01.pdf

[6] M. Schuchmann, M. Rasguljajew, 2013. "Approximation of Non L2(R) Functions on a Compact Interval with a Wavelet Base". Journal of Approximation Theory and Applied Mathematics, Vol. 2.

M. Schuchmann, M. Rasguljajew, 2013. "Implementation and Testing an Algorithm for a Wavelet Collocation Method in Mathematica". International Journal of Emerging Technology and Advanced Engineering, Vol. 3, Issue 6, June 2013. http://www.ijetae.com/files/Volume3Issue6/IJETAE_0613_01.pdf